Homework 5; Due 8.30 a.m. Wed Feb 11. Ross Problems: Ch 6: 23, 27 (b), 42(a), 8, 43

23. $f_{X,Y}(x,y) = 12xy(1-x)$ on $0<x<1$, $0<y<1$ and 0 otherwise.

(a) So $f_{X,Y}(x,y) = g_1(x)g_2(y)$ for all x, y where

$g_1(x) = 6x(1-x)$ on $0<x<1$ and 0 otherwise, $g_2(y) = 2y$ on $0<y<1$ and 0 otherwise

and this is the right way to factorize the constant as each of g_1 and g_2 integrates to 1.

So yes they are independent. (and $f_X(x) = 6x(1-x)$ on $0<x<1$).

(b) $\mathrm{E}(X) = \int x\ f_X(x)\ dx = \int_0^1 (6x^2 - 6x^3)dx = 2 - 1.5 = 0.5$. (Or cite symmetry about $x = 1/2$).

(c) Fron (a), $f_Y(y) = 2y$ on $0<y<1$. $\mathrm{E}(y) = \int_0^1 2y^2 dy = 2/3$.

(d) $\mathrm{E}(X^2) = \int_0^1 (6x^3 - 6x^4)dx = (6/4) - (6/5) = 6/20$. $\mathrm{var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = 6/20 - 1/4 = 1/20$.

(e) $\mathrm{E}(Y^2) = \int_0^1 2y^3 dy = 1/2$, $\mathrm{var}(Y) = \mathrm{E}(Y^2) - (\mathrm{E}(Y))^2 = (1/2) - (4/9) = 1/18$.

27(b) $X \sim U(0,1)$, $Y \sim \mathcal{E}(1)$, X, Y independent. $f_{X,Y}(x,y) = \exp(-y)$ on $0<x<1$, $0<y<\infty$ and 0 otherwise.

$$P(W \equiv X/Y \le w) = \int_{x=0}^{1}\int_{y=x/w}^{\infty} \exp(-y)dydx = \int_{x=0}^{1} \exp(-x/w)dx$$
$$= [-w\exp(-x/w)]_0^1 = w(1-\exp(-1/w)) \quad 0<w<\infty$$

Note: If we integrate X first, then $x \le \min(yw, 1)$ which gets messy.

42(a) $f_{X,Y}(x,y) = x\exp(-x(y+1))$ on $0<x<\infty$, $0<y<\infty$.

$$\begin{aligned}
f_X(x) &= \int_{y=0}^{\infty} xe^{-x}e^{-xy}dy = xe^{-x}(1/x) = \exp(-x)\\
f_Y(y) &= \int_{x=0}^{\infty} x\ \exp(-x(y+1))\ dx = (y+1)^{-2}\int_{w=0}^{\infty} w\ \exp(-w)\ dw = 1/(1+y)^2\\
f_{Y|X}(y|X=x) &= f_{X,Y}(x,y)/f_X(x) = x\exp(-xy) \quad \text{on } 0<y<\infty\\
f_{X|Y}(x|Y=y) &= f_{X,Y}(x,y)/f_Y(y) = (y+1)^2 x\exp(-x(y+1)) \quad \text{on } 0<x<\infty
\end{aligned}$$

Note, $Y|X=x$ is exponential with rate parameter x: $(Y|X=x) \sim \mathcal{E}(x)$.

Note, $X|Y=y$ is Gamma with shape 2 and rate parameter $y+1$: $(X|Y=y) \sim G(2, y+1)$.

8. $f_{X,Y}(x,y) = c(y^2-x^2)\exp(-y)$, $-y<x<y$, $0<y<\infty$

(a)
$$f_Y(y) = c\exp(-y)\int_{-y}^{y}(y^2-x^2)\ dx = c\exp(-y)(2y^3 - 2y^3/3) = (4/3)cy^3\exp(-y).$$
$$1 = \int_0^\infty f_Y(y)dy = (4/3)c\Gamma(4) = (4/3)c(3!) = 8c \quad \text{So} \quad c = 1/8$$

(b) So from (a) $f_Y(y) = (y^3/6)\exp(-y)$ on $0<y<\infty$ and 0 otherwise. Have symmetry in x so let $x>0$

$$f_X(x) = (1/8)\int_x^\infty (y^2-x^2)\exp(-y)dx$$
$$\int y^2e^{-y}dy = -y^2e^{-y} + \int 2ye^{-y}dy = -y^2e^{-y} - 2ye^{-y} + 2\int e^{-y}dy = -e^{-y}(y^2 + 2y + 2)$$
$$f_X(x) = (1/8)\exp(-x)(x^2+2x+2-x^2) = (1/4)(x+1)\exp(-x)$$

Hence by symmetry, $f_X(x) = (1/4)(1+|x|)\exp(-|x|)$ for all x.

43 We have done all the work in #8, but note X and Y are reversed in this question relative to #8. So now

$$\begin{aligned}
f_{X,Y}(x,y) &= (1/8)(x^2-y^2)\exp(-x), \quad -x<y<x, \quad 0<x<\infty\\
f_X(x) &= (1/6)x^3\exp(-x) \quad \text{from question \#8}\\
f_{Y|X}(y|X=x) &= f_{X,Y}(x,y)/f_X(x) = (1/8)(x^2-y^2)\exp(-x)I(-x\le y\le x)/(1/6)x^3\exp(-x)\\
&= (3/4)(1/x)(1 - (y/x)^2)I(-x\le y\le x)
\end{aligned}$$

Note: this is a valid density: it integrates to 1 for every x; also book soln is CDF, but this is sufficient!.

Also we do not need to know $c = 1/8$ – it will cancel out anyhow.