Homework 2; Due 8.30 a.m. Wed January 21: Ross Problems: Ch 5: 1, 8, 11, 32. Ch 5: TE 8.

Ch 5: 1, (a)

$$1 \;=\; \int_{-1}^{1} c(1-x^2)\;dx \;=\; c[x - x^3/3]_{-1}^{1} \;=\; c\;((1-1/3) - (-1-(-1/3))) \;=\; 4c/3$$

So $c = 3/4$.

(b)

$$F_X(x) \;=\; \int_{-1}^{x} (3/4)(1-z^2)\;dz \;=\; (3/4)[z - z^3/3]_{-1}^{x} \;=\; (3/4)(x - x^3/3 + 2/3) \quad \text{for } -1 \le x \le 1$$

and $F_X(x) = 0$ if $x \le -1$, and $F_X(x) = 1$ if $x \ge 1$.

Ch 5: 8,

$$\mathrm{E}(X) \;=\; \int_0^\infty x\; f_X(x)\; dx \;=\; \int_0^\infty x^2 \exp(-x)dx \;=\; \Gamma(3) \;=\; 2! \;=\; 2$$

Or, integrating by parts:

$$\mathrm{E}(X) \;=\; \int_0^\infty x^2 \exp(-x)dx \;=\; [-x^2\exp(-x)]_0^\infty \;+\; \int_0^\infty 2x\exp(-x)dx \;=\; 2\int_0^\infty f_X(x)dx \;=\; 2$$

since the density must integrate to 1 (or do it by parts again).

Ch 5: 11, The position of the point X is uniform on $(0, L)$: $f_X(x) = 1/L$ on $0 < x < L$. The lengths of the two pieces are X and $L-X$, are we are interested in $Y = X/(L-X)$, and $P(Y < 1/4) + P(Y > 4) = 2P(Y < 1/4)$ by symmetry: draw a picture!.

Now $P(Y < 1/4) \;=\; P(4X < (L-X)) \;=\; P(X < L/5) \;=\; 1/5$, so the required probability is 2/5 or 0.4.

Ch 5: 32. $f_T(t) \;=\; \frac{1}{2}\exp(-t/2)$ on $0 < t < \infty$. $P(T > s) \;=\; \int_s^\infty f_T(t)\;dt \;=\; [-\exp(-t/20]_s^\infty \;=\; \exp(-s/2)$.

(a) $P(T > 2) \;=\; \exp(-1) \;=\; 0.3679$.

(b) $P(T > 10 \mid T > 9) \;=\; P(T > 10)/P(T > 9) \;=\; \exp(-5)/\exp(-4.5) \;=\; \exp(-0.5) \;=\; 0.6065$.

(Or, just the same as $P(T > 1)$, by forgetting property of exponential.)

Ch 5: TE 8. X does not have to be continuous, but easiest to write it this way:

$\mathrm{E}(X^2) \;=\; \int x^2\; f_X(x)\; dx \;=\; \int_0^c x\; x\; f_X(x)\; dx \;\le\; c\int_0^c x f_X(x)dx \;=\; c\mathrm{E}(X)$.

Note that, by same argument, $0 \le \mathrm{E}(X) \le c$, so let $\mathrm{E}(X) = \alpha\; c$ for some number α, $0 < \alpha < 1$.

So now $\mathrm{var}(X) \;=\; \mathrm{E}(X^2) - (\mathrm{E}(X))^2 \;\le\; c(\alpha\; c) \;-\; (\alpha\; c)^2 \;=\; c\;\alpha(1-\alpha)$.

But now max of $\alpha(1-\alpha)$ over $0 < \alpha < 1$ is 1/4 (at $\alpha = \frac{1}{2}$).

So then $\mathrm{var}(X) \le c^2/4$.

(Note, in fact we can only get this variance when $P(X = 0) = P(X = c) = \frac{1}{2}$.)