Explain/justify all your your answers.

1. *12 points total: 3 each part*

$f_{X,Y}(x,y) = 2$ on $0 \le x,\ 0\ \le y,\ (x+y) \le 1$ and $f_{X,Y}(x,y) = 0$ for all other (x,y).

(a) $f_X(x) \ = \ \int_0^{1-x} 2\ dy \ = \ 2(1-x)$ on $0 < x < 1$.

(b) $\mathrm{E}(X) \ = \ \int_0^1 2x(1-x)\ dx \ = \ 1/3$,

$\mathrm{E}(X^2) \ = \ \int_0^1 2x^2(1-x)\ dx \ = \ 1/6$; $\mathrm{var}(X) = 1/6 - 1/9 \ = \ 1/18$.

(c) $\mathrm{E}(XY) \ = \ \int\int_{x+y\le 1} 2xy`dx\ dy \ = \ 2\int_0^1 x\left(\int_0^{1-x} y\ dy\right) dx \ = \ 1/12$

(d) $\rho(X,Y) \ = \ \mathrm{cov}(X,Y)/\sqrt{var(X).var(Y)} \ = \ (1/12 - (1/3)^2)/(1/18) \ = \ -(1/36)/(1/18) \ = \ -1/2$.

2. *12 points total; 3 each part*

Suppose Z_1, Z_2 and Z_3 are independent standard Normal random variables with mean 0 and variance 1.

Let $X = 3 + 3Z_1 + 4Z_2$ and $Y = 6 - 2Z_1 + 3Z_3$.

(a) $\mathrm{E}(X) \ = \ 3+0+0 \ = \ 3$. $\mathrm{var}(X) \ = \ 3^2 + 4^2 \ = \ 25$.

$\mathrm{E}(Y) \ = \ 6$. $\mathrm{var}(Y) \ = \ (-2)^2 + 3^2 \ = \ 13$.

(b) $\mathrm{cov}(X,Y) \ = \ cov(3Z_1 + 4Z_2, -2Z_1 + 3Z_3) \ = \ -6\mathrm{var}(Z_1) \ = \ -6$,

(c) $\mathrm{E}(Y-2X) \ = \ 6-2\times 3 \ = \ 0$. $\mathrm{var}(Y-2X) \ = \ \mathrm{var}(Y)+4\mathrm{var}(X)-4\mathrm{cov}(X,Y) \ = \ 13+4\times 25-4\times(-6) \ = \ 137$.

(d) $P(2Y+1 \ = \ 4X+1) \ = \ P(Y-2X \ > \ 0) \ = \ 1/2$, since from (c) $(Y-2X)$ is Normal with mean 0.

3. *12 points total; 3 each part*

Let $X_i = 1$ if both bulbs of the ith pair are red, and $X_i = 0$ otherwise $i = 1, 2, ..., 20$.

(a) $\mathrm{E}(X_i) \ = \ P(X_1 = 1) \ = \ (20/40).(19/39) \ = \ 19/78$. $\mathrm{var}(X_i) \ = \ (19/78).(1 - 19/78) \ = \ 0.184$.

(b) $\mathrm{E}(X_iX_j) \ = \ P(X_i = X_j = 1) \ = \ P(4\text{ red}) \ = \ (20/40).(19/39).(18.38).(17/37) \ = \ 0.0530$.

$\mathrm{cov}(X_i, X_j) = 0.0530 - (19/78)^2 \ = \ -0.0063$.

(c) Expected number of pairs for which both tulips are red is $20 * \mathrm{E}(X_i) \ = \ 4.87$.

(d) By symmetry, expected pairs of white tulips is also 4.87.

So expected number of red-white pairs is $20 - 2\times 4.87 \ = \ 10.256$.