MATH/STAT 395 Midterm-2: Winter 2009 Feb 27, 2009
Useful facts

1. Permutations and combinations

There are $n! = \prod_{i=1}^{n} i = 1.2.3.4.\dots n$ permutations of n objects.

There are $\binom{n}{k} = n!/(k!(n-k)!)$ ways of choosing a given k objects from n.

2. Joint and conditional probabilities

If C and D are any events: $\mathrm{P}(C \bigcup D) = \mathrm{P}(C) + \mathrm{P}(D) - \mathrm{P}(C \bigcap D)$.

The conditional probability of C given D is $\mathrm{P}(C \mid D) = \mathrm{P}(C \bigcap D) / \mathrm{P}(D)$.

C and D are independent if $\mathrm{P}(C \bigcap D) = \mathrm{P}(C).\mathrm{P}(D)$.

3. Laws and theorems

Suppose $E_1, \dots, E_k$ is a partition of Ω.

That is $E_i \bigcap E_j$ is empty, and $E_1 \bigcup E_2 \bigcup \dots \bigcup E_k = \Omega$.

The law of total probability states that: $\mathrm{P}(D) = \sum_{j=1}^{k} \mathrm{P}(D \bigcap E_j) = \sum_{j=1}^{k} \mathrm{P}(D \mid E_j)\ \mathrm{P}(E_j)$

Bayes' Theorem states that: $\mathrm{P}(E_i \mid D) = \mathrm{P}(D \mid E_i)\ \mathrm{P}(E_i)/\mathrm{P}(D)$

4. Random variables and distributions

	discrete (mass)	continuous (density)
Probability mass/density function	pmf: $\mathrm{P}(X=x) = p_X(x)$	pdf: $f_X(x)$
Cumulative dist. func. CDF, $\mathrm{P}(X \le x)$	$F_X(x) = \sum_{w \le x} p_X(w)$	$F_X(x) = \int_{-\infty}^{x} f_X(w)dw$
Joint mass/density func. of (X,Y)	$p_{X,Y}(x,y) = \mathrm{P}(X=x, Y=y)$	$f_{X,Y}(x,y)$
Marginal mass/density of X	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{y=-\infty}^{\infty} f_{X,Y}(x,y)dy$
Conditional of X given $Y=y$	$p_X(x \mid Y=y) = p_{X,Y}(x,y)/p_Y(y)$	$f_X(x \mid Y=y) = f_{X,Y}(x,y)/f_Y(y)$
Independence of X and Y	$p_{X,Y}(x,y) = \mathrm{P}(X=x)\mathrm{P}(Y=y)$	$f_{X,Y}(x,y) = f_X(x)f_Y(y)$

5. Moments of random variables:

Expectation:	$\mathrm{E}(X) = \sum_x x\ \mathrm{P}(X=x)$	$\int_{-\infty}^{\infty} x f_X(x)dx$
	$\mathrm{E}(g(X)) = \sum_x g(x)\ \mathrm{P}(X=x)$	$\int_{-\infty}^{\infty} g(x) f_X(x)dx$

provided the sum/integral converges absolutely.

For any random variables X:

Variance: $\mathrm{var}(X) = \mathrm{E}((X - E(X))^2) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

Note: $\mathrm{E}(aX+b) = a\mathrm{E}(X) + b$, $\mathrm{var}(aX+b) = a^2\ \mathrm{var}(X)$.

For any random variables X, Y, Z and W:

Covariance: $\mathrm{cov}(X,Y) = \mathrm{E}((X - E(X))(Y - E(Y))) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)$

Correlation: $\rho(X,Y) = \mathrm{cov}(X,Y)/\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}$, $-1 \le \rho(X,Y) \le 1$

Note: $\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y)$, $\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y) + 2\mathrm{cov}(X,Y)$

$\mathrm{cov}(aX+b, cW+d) = ac\,\mathrm{cov}(X,W)$, $\mathrm{cov}(X+Y, W+Z) = \mathrm{cov}(X,W) + \mathrm{cov}(X,Z) + \mathrm{cov}(Y,W) + \mathrm{cov}(Y,Z)$

6. A note about Normal (Gaussian) random variables

(a) Linear transformations of Normal random variables are Normal

(b) Linear combinations of independent Normal r.vs are Normal

(c) Different linear combinations of independent Normal r.vs are called *jointly Normal*

(d) If X and Y are jointly Normal and $\mathrm{cov}(X,Y)$=0, then X and Y are independent.

7. Standard distributions:

	pmf or pdf	mean	variance
(a) Binomial; $B(n,p)$	$\mathrm{P}(X=k) = \binom{n}{k} p^k(1-p)^{n-k}$	np	$np(1-p)$
index n, parameter p	$k=0,1,2,...,n$		
(b) Geometric; $G(p)$;	$\mathrm{P}(X=k) = p(1-p)^{k-1}$	$1/p$	$(1-p)/p^2$
parameter p	$k=1,2,3,4,......$		
(c) Neg. Binomial; $NegB(r,p)$;	$\mathrm{P}(X=k) = \binom{k-1}{r-1} p^r(1-p)^{k-r}$	r/p	$r(1-p)/p^2$
index r, parameter p	$k=r,r+1,r+2,....$		
(d) Poisson; $\mathcal{P}o(\mu)$	$\mathrm{P}(X=k) = \exp(-\mu)\mu^k/k!, \quad k=0,1,2,...$	μ	μ
(e) Multinomial, $Mn(n,(p_1,\ldots,p_k))$— just like binomial but with k outcome types			
(f) Uniform on (a,b); $U(a,b)$;	$f_X(x)=1/(b-a), \quad a<x<b$	$(b+a)/2$	$(b-a)^2/12$
(g) Normal, $N(\mu,\sigma^2)$	$f_X(x)=(1/\sqrt{2\pi\sigma^2})\exp(-(x-\mu)^2/2\sigma^2)$	μ	σ^2
(h) Exponential, $\mathcal{E}(\lambda)$	$f_X(x)=\lambda\exp(-\lambda x)$	$1/\lambda$	$1/\lambda^2$
rate parameter λ	$0\le x<\infty$		
(i) Gamma, $G(\alpha,\lambda)$	$f_X(x)=\lambda^\alpha x^{\alpha-1}\exp(-\lambda x)/\Gamma(\alpha)$	α/λ	α/λ^2
shape α, rate λ	$0\le x<\infty$		

Note: $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}\exp(-x)dx$ $\quad\Gamma(\alpha)=(\alpha-1)\Gamma(\alpha-1)$, and $\Gamma(n)=(n-1)!$ for integer n.

8. Moment generating functions: $M_X(t) = \exp(tX)$

	mgf	note
(a) Binomial; $B(n,p)$	$(q+p\ z))^n$	where $q=1-p$ and $z\equiv\exp(t)$
(b) Geometric; $G(p)$;	$pz/(1-q\ z)$	where $q=1-p$ and $z\equiv\exp(t)$
(c) Neg. Binomial; $NegB(r,p)$;	$(pz/(1-q\ z))^r$	where $q=1-p$ and $z\equiv\exp(t)$
(d) Poisson; $\mathcal{P}o(\mu)$	$\exp(\mu(z-1))$	where $z\equiv\exp(t)$
μ		
(g) Normal, $N(\mu,\sigma^2)$	$\exp(\mu\ t+\frac{1}{2}\sigma^2t^2)$	
(h) Exponential, $\mathcal{E}(\lambda)$	$\lambda/(\lambda-t)$	provided $t < \lambda$
(i) Gamma, $G(\alpha,\lambda)$	$(\lambda/(\lambda-t))^\alpha$	provided $t < \lambda$

(j) For a bivariate Normal density:

$M_{X,Y}(s,t) = \mathrm{E}(\exp(sX+tY)) = \exp(\mu\ s+\nu\ t+\frac{1}{2}\sigma^2s^2 + \frac{1}{2}\tau^2t^2+\rho\ \sigma\ \tau\ s\ t)$

where $X \sim N(\mu,\sigma^2)$, $Y \sim N(\nu,\tau^2)$, and ρ is the correlation between X and Y.