

Useful facts

**1. Permutations and combinations**

There are  $n! = \prod_{i=1}^n i = 1.2.3.4. \dots n$  permutations of  $n$  objects.

There are  $\binom{n}{k} = n!/(k!(n-k)!)$  ways of choosing a given  $k$  objects from  $n$ .

**2. Joint and conditional probabilities**

If  $C$  and  $D$  are any events:  $P(C \cup D) = P(C) + P(D) - P(C \cap D)$ .

The conditional probability of  $C$  given  $D$  is  $P(C | D) = P(C \cap D) / P(D)$ .

$C$  and  $D$  are independent if  $P(C \cap D) = P(C).P(D)$ .

**3. Laws and theorems**

Suppose  $E_1, \dots, E_k$  is a partition of  $\Omega$ .

That is  $E_i \cap E_j$  is empty, and  $E_1 \cup E_2 \cup \dots \cup E_k = \Omega$ .

The law of total probability states that:  $P(D) = \sum_{j=1}^k P(D \cap E_j) = \sum_{j=1}^k P(D | E_j) P(E_j)$

Bayes' Theorem states that:  $P(E_i | D) = P(D | E_i) P(E_i)/P(D)$

**4. Random variables and distributions**

	discrete (mass)	continuous (density)
Probability mass/density function	pmf: $P(X = x) = p_X(x)$	pdf: $f_X(x)$
Cumulative dist. func. CDF, $P(X \leq x)$	$F_X(x) = \sum_{w \leq x} p_X(w)$	$F_X(x) = \int_{-\infty}^x f_X(w)dw$
Joint mass/density func. of $(X, Y)$	$p_{X,Y}(x, y) = P(X = x, Y = y)$	$f_{X,Y}(x, y)$
Marginal mass/density of $X$	$p_X(x) = \sum_y p_{X,Y}(x, y)$	$f_X(x) = \int_{y=-\infty}^{\infty} f_{X,Y}(x, y)dy$
Conditional of $X$ given $Y = y$	$p_X(x Y = y) = p_{X,Y}(x, y)/p_Y(y)$	$f_X(x Y = y) = f_{X,Y}(x, y)/f_Y(y)$
Independence of $X$ and $Y$	$p_{X,Y}(x, y) = P(X = x)P(Y = y)$	$f_{X,Y}(x, y) = f_X(x)f_Y(y)$

**5. Moments of random variables:**

Expectation:  $E(X) = \sum_x x P(X = x)$   $\int_{-\infty}^{\infty} x f_X(x)dx$   
 $E(g(X)) = \sum_x g(x) P(X = x)$   $\int_{-\infty}^{\infty} g(x) f_X(x)dx$

provided the sum/integral converges absolutely.

**For any random variables  $X$ :**

Variance:  $\text{var}(X) = E((X - E(X))^2) = E(X^2) - (E(X))^2$

Note:  $E(aX + b) = aE(X) + b$ ,  $\text{var}(aX + b) = a^2 \text{var}(X)$ .

**For any random variables  $X, Y, Z$  and  $W$ :**

Covariance:  $\text{cov}(X, Y) = E((X - E(X))(Y - E(Y))) = E(XY) - E(X)E(Y)$

Correlation:  $\rho(X, Y) = \text{cov}(X, Y) / \sqrt{\text{var}(X)\text{var}(Y)}$ ,  $-1 \leq \rho(X, Y) \leq 1$

Note:  $E(X + Y) = E(X) + E(Y)$ ,  $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2\text{cov}(X, Y)$

$\text{cov}(aX + b, cW + d) = ac \text{cov}(X, W)$ ,  $\text{cov}(X + Y, W + Z) = \text{cov}(X, W) + \text{cov}(X, Z) + \text{cov}(Y, W) + \text{cov}(Y, Z)$

$E(X) = E(E(X | Y))$ ,  $\text{var}(X) = \text{var}(E(X|Y)) + E(\text{var}(X|Y))$

**6. A note about Normal (Gaussian) random variables**

- (a) Linear transformations of Normal random variables are Normal
- (b) Linear combinations of independent Normal r.vs are Normal
- (c) Different linear combinations of independent Normal r.vs are called *jointly Normal*
- (d) If  $X$  and  $Y$  are jointly Normal and  $\text{cov}(X, Y)=0$ , then  $X$  and  $Y$  are independent.

## 7. Standard distributions:

	pmf or pdf	mean	variance
(a) Binomial; $B(n, p)$ index $n$ , parameter $p$	$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ $k = 0, 1, 2, \dots, n$	$np$	$np(1-p)$
(b) Geometric; $G(p)$ parameter $p$	$P(X = k) = p(1-p)^{k-1}$ $k = 1, 2, 3, 4, \dots$	$1/p$	$(1-p)/p^2$
(c) Neg. Binomial; $NegB(r, p)$ index $r$ , parameter $p$	$P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$ $k = r, r+1, r+2, \dots$	$r/p$	$r(1-p)/p^2$
(d) Poisson; $\mathcal{P}o(\mu)$	$P(X = k) = \exp(-\mu)\mu^k/k!, \quad k = 0, 1, 2, \dots$	$\mu$	$\mu$
(e) Multinomial, $Mn(n, (p_1, \dots, p_k))$ — just like binomial but with $k$ outcome types			
(f) Uniform on $(a, b)$ ; $U(a, b)$ ;	$f_X(x) = 1/(b-a), \quad a < x < b$	$(b+a)/2$	$(b-a)^2/12$
(g) Normal, $N(\mu, \sigma^2)$	$f_X(x) = (1/\sqrt{2\pi\sigma^2}) \exp(-(x-\mu)^2/2\sigma^2)$	$\mu$	$\sigma^2$
(h) Exponential, $\mathcal{E}(\lambda)$ rate parameter $\lambda$	$f_X(x) = \lambda \exp(-\lambda x)$ $0 \leq x < \infty$	$1/\lambda$	$1/\lambda^2$
(i) Gamma, $G(\alpha, \lambda)$ shape $\alpha$ , rate $\lambda$	$f_X(x) = \lambda^\alpha x^{\alpha-1} \exp(-\lambda x)/\Gamma(\alpha)$ $0 \leq x < \infty$	$\alpha/\lambda$	$\alpha/\lambda^2$
Note: $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \exp(-x) dx$ $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$ , and $\Gamma(n) = (n-1)!$ for integer $n$ .			

## 8. Moment generating functions: $M_X(t) = \exp(tX)$

	mgf	note
(a) Binomial; $B(n, p)$	$(q + pz)^n$	where $q = 1-p$ and $z \equiv \exp(t)$
(b) Geometric; $G(p)$ ;	$pz/(1-qz)$	where $q = 1-p$ and $z \equiv \exp(t)$
(c) Neg. Binomial; $NegB(r, p)$ ;	$(pz/(1-qz))^r$	where $q = 1-p$ and $z \equiv \exp(t)$
(d) Poisson; $\mathcal{P}o(\mu)$	$\exp(\mu(z-1))$	where $z \equiv \exp(t)$
(g) Normal, $N(\mu, \sigma^2)$	$\exp(\mu t + \frac{1}{2}\sigma^2 t^2)$	
(h) Exponential, $\mathcal{E}(\lambda)$	$\lambda/(\lambda-t)$	provided $t < \lambda$
(i) Gamma, $G(\alpha, \lambda)$	$(\lambda/(\lambda-t))^\alpha$	provided $t < \lambda$
(j) For a bivariate Normal density:		

$M_{X,Y}(s, t) = E(\exp(sX + tY)) = \exp(\mu s + \nu t + \frac{1}{2}\sigma^2 s^2 + \frac{1}{2}\tau^2 t^2 + \rho \sigma \tau s t)$   
 where  $X \sim N(\mu, \sigma^2)$ ,  $Y \sim N(\nu, \tau^2)$ , and  $\rho$  is the correlation between  $X$  and  $Y$ .