Lecture 19: More expectations. Summing independent random variables

(The general case is in Ross Chapter 7.

Here we take the simplest case of two discrete random variables, but the results are true in general.)

19.1 Independent random variables

Let X and Y be two discrete random variables defined on the same sample space. (e.g. Ω is outcomes of tosses of two dice; X is the sum of the two values, Y is the maximum of the two values.)

Suppose X takes values x_i, $i = 1, 2, ...$, and Y takes values y_j, j=1,2,3,....

Let $P(X = x_i \ \cap \ Y = y_j) \ = \ p_{ij}$. Note $\sum_i \sum_j p_{ij} \ = \ 1$.

Also $P(X = x_i) \ = \ \sum_j P(X = x_i \ \cap \ Y = y_j) \ = \ \sum_j p_{ij}$

and $P(Y = y_j) \ = \ \sum_i P(X = x_i \ \cap \ Y = y_j) \ = \ \sum_i p_{ij}$.

Also note that *events* of interest are of the form $(X = x_i \ \cap \ Y = y_j)$; alll other statements about X and Y derive from these.

Definition: Random variables X and Y are **independent** if

$$P(X = x_i \ \cap \ Y = y_j) \ = \ P(X = x_i) \times P(Y = y_j) \text{ for all } i \text{ and } j.$$

19.2 Expectation of the sum (Note: this does not require independence.)

The set of possible values of $X + Y$ is the set of all $x_i + y_j$ for all i and j:

$$\begin{aligned}
\mathrm{E}(X+Y) &= \sum_i \sum_j (x_i + y_j) P(X = x_i \ \cap \ Y = y_j) \ = \ \sum_i \sum_j x_i p_{ij} \ + \ \sum_i \sum_j y_j p_{ij} \\
&= \sum_i (x_i \sum_j p_{ij}) \ + \ \sum_j (y_j \sum_i p_{ij}) \ = \ \sum_i x_i P(X = x_i) \ + \ \sum_j y_j P(Y = y_j) \\
&= \mathrm{E}(X) \ + \ \mathrm{E}(Y) \quad \textbf{Expectation of sum is sum of expectations: always}.
\end{aligned}$$

19.3 Expectation of the product: independence case

In general: $\mathrm{E}(XY) \ = \ \sum_i \sum_j x_i y_j P(X = x_i \ \cap \ Y = y_j) \ = \ \sum_i \sum_j x_i y_j p_{ij}$.

If X and Y are **independent** then

$$\begin{aligned}
\mathrm{E}(XY) &= \sum_i \sum_j x_i y_j p_{ij} \ = \ \sum_i \sum_j x_i y_j P(X = x_i) P(Y = y_j) \\
&= \left(\sum_i x_i P(X = x_i) \right) \left(\sum_j y_j P(Y = y_j) \right) \ = \ \mathrm{E}(X)\mathrm{E}(Y).
\end{aligned}$$

19.4 Variance of the sum: independence case

$$\begin{aligned}
\mathrm{var}(X+Y) &= \mathrm{E}((X+Y)^2) \ - \ (\mathrm{E}(X+Y))^2 \\
\mathrm{E}((X+Y)^2) &= \mathrm{E}(X^2 + 2XY + Y^2) \ = \ \mathrm{E}(X^2) + 2\mathrm{E}(XY) + \mathrm{E}(Y^2) \\
(\mathrm{E}(X+Y))^2 &= (\mathrm{E}(X) + \mathrm{E}(Y))^2 \ = \ (\mathrm{E}(X))^2 + 2\mathrm{E}(X)\mathrm{E}(Y) + (\mathrm{E}(Y))^2 \\
\mathrm{var}(X+Y) &= \mathrm{var}(X) \ + \ 2(\mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)) \ + \ \mathrm{var}(Y)
\end{aligned}$$

If X and Y are **independent**, $\mathrm{E}(XY) \ = \ \mathrm{E}(X).\mathrm{E}(Y)$. Then:

$$\mathrm{var}(X+Y) \ = \ \mathrm{var}(X) \ + \ \mathrm{var}(Y)$$

For independent random variables, the variance of the sum is the sum of the variances.

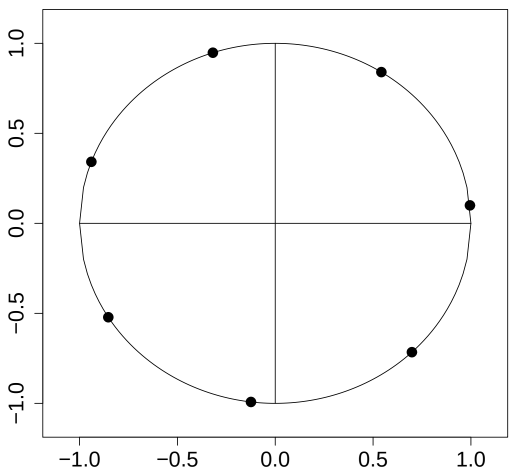
Note 1: If we can sum 2, we can sum any finite number: $X + Y + Z \ = \ (X + Y) \ + \ Z$.

Note 2: The converse of 19.3, 19.4 is NOT true.

We can have $\mathrm{E}(XY) \ = \ \mathrm{E}(X).\mathrm{E}(Y)$, but X and Y NOT **independent**.

Lecture 20: Examples of computing expectations

20.1 An example of variance of sum



(a) What are $\mathrm{E}(X)$, $\mathrm{E}(Y)$, $\mathrm{E}(XY)$?

(b) Are X and Y independent?

(c) What are $\mathrm{var}(X)$, $\mathrm{var}(X+Y)$ and $\mathrm{var}(X^2+Y^2)$?

X	0.5423	-0.3187	-0.9398	-0.8532	-0.1241	0.6984	0.9950	0
Y	0.8402	0.9478	0.3418	-0.5215	-0.9923	-0.7157	0.0998	0
XY	0.4556	-0.3021	-0.3212	0.4450	0.1231	-0.4998	0.0993	0
X^2	0.2941	0.1016	0.8832	0.7279	0.0154	0.4878	0.9900	3.5
X^2+Y^2	1.0	1.0	1.0	1.0	1.0	1.0	1.0	7.0
prob	1/7	1/7	1/7	1/7	1/7	1/7	1/7	1

20.2 Example of expectations for a continuous random variable

Suppose X is a Uniform (0,1) random variable: $f(x) = 1,\ 0 \le x \le 1$.

(a) What are $\mathrm{E}(X)$, $\mathrm{E}(X^2)$ and $\mathrm{E}(X^n)$?

(b) What is $\mathrm{var}(X)$?

(c) Now suppose Y is a Uniform (a, b) random variable: $f_Y(y) = 1/(b-a)$ on $a \le y \le b$.

Without doing any more integrals: what are $\mathrm{E}(Y)$ and $\mathrm{var}(Y)$?

20.3 Using the Poisson approximation on the "birthday problem"

We know that doing the exact calculation we need only $n = 23$ people before the probability that no two share a birthday is less than 1/2.

(a) Find out whether the Poisson approximation for independent rare events given the same result – i.e. how large must n be before the probability of 0 sharing events is less than 1/2.

(b) In our sample of 37 people, we did not have any trios sharing a birthday. How would you use the Poisson approximation, and the assumption of approximately independent events, to find the probability of this?

(c) How would you then use this same Poisson approximation to find the smallest value of n for which this probability is less than 1/2.

Lecture 21: Midterm -2 crib sheet; suggested review Exx on web site.

1. Permutations and combinations

There are $n! = \prod_{i=1}^{n} i = 1.2.3.4.\ldots.n$ permutations of n objects.

There are $\binom{n}{k} = n!/(k!(n-k)!)$ ways of choosing a given k objects from n.

2. Joint and conditional probabilities

If C and D are any events: $P(C \bigcup D) = P(C) + P(D) - P(C \bigcap D)$.

The conditional probability of C given D is $P(C \mid D) = P(C \bigcap D) / P(D)$.

C and D are independent if $P(C \bigcap D) = P(C).P(D)$.

3. Laws and theorems

Suppose $E_1, \ldots, E_k$ is a partition of Ω.

That is $E_i \bigcap E_j$ is empty, and $E_1 \bigcup E_2 \bigcup \ldots \bigcup E_k = \Omega$.

The law of total probability states that: $P(D) = \sum_{j=1}^{k} P(D \bigcap E_j) = \sum_{j=1}^{k} P(D \mid E_j)\ P(E_j)$

Bayes' Theorem states that: $P(E_i \mid D) = P(D \mid E_i)\ P(E_i)/P(D)$

4. Random variables and distributions

	discrete (mass)	continuous (density)
Probability mass/density function	pmf: $P(X=x) = p_X(x)$	pdf: $f_X(x)$
defined for	all x with $p_X(x) > 0$	$-\infty < x < \infty$
Expectation E(X)	$\sum_x x p_X(x)$	$\int_{-\infty}^{\infty} x f_X(x) dx$
Result: E($g(X)$)	$\sum_x g(x) p_X(x)$	$\int_{-\infty}^{\infty} g(x) f_X(x) dx$

Variance: $\mathrm{E}((X - (\mathrm{E}(X)))^2) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

5. Standard distributions: add means/variances!!

p_X or f_X	values	mean E(X)	variance
(a) Binomial; $B(n,p)$; index n, parameter p;			
$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$	$k = 0, 1, 2, ..., n$	np	$np(1-p)$
(b) Geometric; $G(p)$; parameter p;			
$P(X=k) = p(1-p)^{k-1}$	$k = 1, 2, 3, 4, ...$	$1/p$	$(1-p)/p^2$
(c) Negative Binomial; $NegB(r,p)$			
$P(X=k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$	$k = r, r+1, r+2, ...$	r/p	$r(1-p)/p^2$
(d) Poisson; $Po(\mu)$; parameter μ;			
$P(X=k) = \exp(-\mu)\mu^k/k!$	$k = 0, 1, 2, 3, ...$	μ	μ
(e) ** Uniform on interval (a,b); $U(a,b)$;			
$f_X(x) = 1/(b-a)$	$a < x < b$	$(a+b)/2$	$(b-a)^2/12$
(f) ** Exponential, $E(\lambda)$, rate parameter λ;			
$f_X(x) = \lambda \exp(-\lambda x)$	$0 \leq x < \infty$	$1/\lambda$	$1/\lambda^2$

**: No continuous random variables in midterm-2.

6. Summation of series:

(a) $\sum_{i=0}^{n-1} x^i = (1 + x + x^2 + ... + x^{n-1}) = (1-x^n)/(1-x)$

(b) $\sum_{i=0}^{\infty} x^i/i! = (1 + x + x^2/2 + x^3/6 + ...) = \exp(x)$