Lecture 16: The Bernoulli process, and associated random variables

16.1: The process

0 0 1 0 0 0 1 1 0 0 1 0 0 0 0 0 0 0 1 0 0 1 0 0 1: Each *trial* is *success* (1) or not (0).

$X_1 X_2 X_3 X_4$.... ...X_{25}...: Each X_i is 0 or 1.

... ... T_5... ... ... T_{10}... ... ... T_{15}... ... ... T_{20}... ... ... T_{25}...: $T_n = X_1 + ... + X_n$

$- - -Y_1 - - - -Y_2 Y_3 - -Y_4 - - - - - - - - - - Y_5 - -Y_6 - - - Y_7$: Y_r is rth inter-arrival time

... ...W_1... ... ...$W_2 W_3$.. ..W_4... ... ...W_5... ...W_6... ...W_7: W_r is total waiting time to rth 1.

The Bernoulli process is defined by X_i independent, with $P(X_i = 1) = p$ and $P(X_i = 0) = (1 - p)$.

$T_n = X_1 + ... + X_n$ is number of *successes* (i.e. 1s) in first n trials.

Y_r is the *inter-arrival time*: number of trials from $(r - 1)$th success to r th.

$W_r = Y_1 + Y_2 + ... + Y_r$ is number of trials to r th. success.

$Y_r^* = Y_r - 1$: number of failures (0) before next success (1).

$W_r^* = Y_1^* + ... + Y_r^*$: number of failures (0) before r th success. **Note:** $W_r > n$ if and only if $T_n < r$.

16.2: Bernoulli and Binomial random variables Ross 4.6

(i) X_i is Bernoulli(p). $P(X_i = 1) = p$, $P(X_i = 0) = (1 - p)$.

$\mathrm{E}(X_i) = p \times 1 + (1 - p) \times 0 = p$

$\mathrm{E}(X_i^2) = p \times 1^2 + (1 - p) \times 0^2 = p$, so $\mathrm{var}(X_i) = \mathrm{E}(X_i^2) - (\mathrm{E}(X_i))^2 = p - p^2 = p(1 - p)$.

$T_n = X_1 + ... + X_n$ is Binomial (n, p).

The probability of each sequence of k 1's and $(n - k)$ 0's is $p^k(1 - p)^{n-k}$ and there are $\binom{n}{k}$ such sequences.

$P(T_n = k) = \binom{n}{k} p^k (1 - p)^{n-k}$.

Expectations always add: see 15.2. $\mathrm{E}(T_n) = \mathrm{E}(X_1) + ... + \mathrm{E}(X_n) = p + p + ... + p = np$.

In general variances do NOT add, but here they do: $\mathrm{var}(T_n) = \mathrm{var}(X_1) + ... + \mathrm{var}(X_n) = np(1 - p)$.

Unfortunately Ross does not let us talk about when variances can be added until Chapter 6.

(Alternative derivations of $\mathrm{E}(T_n)$ and $\mathrm{var}(T_n)$ are given in 18.2.)

16.3: Geometric and Negative Binomial random variables Ross 4.8.1, 4.8.2

Y_r are independent, and have Geometric (p) distribution: $P(Y = k) = (1 - p)^{k-1}p$, for $k = 1, 2, 3, ...$.

$\mathrm{E}(Y) = \sum_{k=1}^{\infty} k(1 - p)^{k-1}p = p/(1 - (1 - p))^2 = 1/p$.

$\mathrm{E}(Y(Y - 1)) = 2p(1 - p)/(1 - (1 - p))^3 = 2(1 - p)/p^2$.

So $\mathrm{var}(Y) = \mathrm{E}(Y(Y - 1) + Y) - (\mathrm{E}(Y))^2 = (2(1 - p)/p^2) + 1/p - 1/p^2 = (1 - p)/p^2$ (see "note").

$Y^* = (Y - 1)$, $P(Y^* = k) = (1 - p)^k p$, for $k = 0, 1, 2, 3....$

$\mathrm{E}(Y_r^*) = \mathrm{E}(Y) - 1 = (1 - p)/p$, $\mathrm{var}(Y^*) = \mathrm{var}(Y)$; see 15.2,15.3.

Note: $\sum_r x^r = 1/(1-x)$, so $\sum_r r x^{r-1} = \frac{d}{dx}\frac{1}{1-x} = 1/(1-x)^2$, $\sum_r r(r-1)x^{r-2} = \frac{d}{dx}\frac{1}{(1-x)^2} = 2/(1-x)^3$.

$W_r = Y_1 + ... + Y_r$. Expectations add, so $\mathrm{E}(W_r) = r/p$.

In fact again the variances add, although Ross will not let us say that yet: $\mathrm{var}(W_r) = r(1 - p)/p^2$.

$P(W_r = k) = P(r-1$ successes in $k-1$ trials, and then success$) = \binom{k-1}{r-1}(1-p)^{k-r}p^{r-1}p$ for $k = r, r+1, ...$

$W_r^* = W_r - r$, $\mathrm{E}(W_r^*) = \mathrm{E}(W_r) - r$, $\mathrm{var}(W_r^*) = \mathrm{var}(W_r)$

$P(W_r^* = k) = P(r - 1$ successes in $r + k - 1$ trials, and then success$) = \binom{r+k-1}{r-1}(1 - p)^k p^{r-1}p$

for $k = 0, 1, 2, 3...$

Lecture 17: Examples of Binomial, Geometric and Negative Binomials

A hypothetical story:

Mendel crossed two plants that were red-flowered, but each had one white-flowered parent. He therefore knew that each offspring plant would have white flowers with probability 1/4, independently of all the others. He planted one offspring seed each morning, and they all grew, and each one flowered the exact same number of days after planting. The first one flowered on June 1, 1865.

1. By June 20, 20 plants had flowered.

(a) How many of these plants are expected to have white flowers?

(b) What is the variance of the number of white-flowered plants?

(c) What is the probability that 5 of the plants had white flowers?

2. On June 8th, Mendel saw that the plant newly flowered that day had white flowers.

(a) What is the expected date of the next white-flowered plant?

(b) What is the probability that the next white-flowered plant flowers June 15 *or later*?

(c) What is the probability the next white-flowered plant flowers on June 13?

3. On June 8th, Mendel saw that the plant newly flowered that day had white flowers.

(a) What is the expected number of red flowers flowering before the next white-flowered plant?

(b) What is the variance of this number?

(c) What is the probability this number is at least 3?

(d) Mendel's assistant reminds him that the plant flowering on June 7 also had white flowers, and says that therefore by "the law of averages" they will probably have to wait longer than four days for the next white-flowered plant. What does Mendel say?

4. On June 8th, Mendel saw that the plant newly flowered that day had white flowers.

(a) What is the expected date of flowering of the 5 th white-flowered plant *after the one on June 8*?

(b) What is the probability the 5 th white-flowered plant *after the one on June 8* flowers on June 28?

Lecture 18: Poisson random variables: approximation to Binomial Ross 4.7

18.1 Reminder of facts about the Poisson distribution

(i) From 9.3: $P(X=j) \;=\; e^{-\lambda}\lambda^j/j!, \quad$ for $j=0,1,2,3,....$

(ii) From 15.4: $\mathrm{E}(X) \;=\; = \sum_{x=0}^{\infty} x e^{-\lambda}\lambda^x/x! \;=\; \sum_{x=1}^{\infty} x e^{-\lambda}\lambda^x/x! \;=\; e^{-\lambda}\lambda \sum_{x=1}^{\infty} \lambda^{x-1}/(x-1)! \;=\; e^{-\lambda}\lambda e^{\lambda} \;=\; \lambda$

Then: $\mathrm{E}(X(X-1)) \;=\; \sum_{x=0}^{\infty} x(x-1)e^{-\lambda}\lambda^x/x! \;=\; e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty} \lambda^{x-2}/(x-2)! \;=\; e^{-\lambda}\lambda^2 e^{\lambda} \;=\; \lambda^2$

So then $\mathrm{var}(X) \;=\; \mathrm{E}(X^2) - (\mathrm{E}(X))^2 \;=\; \mathrm{E}(X(X-1)) \;+\; \mathrm{E}(X) - (\mathrm{E}(X))^2 \;=\; \lambda^2 + \lambda - \lambda^2 \;=\; \lambda$

(iii) Useful model for numbers of things (accidents, hurricanes, centenarians, errors, customers, judicial vacancies, ...crossovers in genetics (see 11.1 (ii))), when there are a very large number of opportunities for the "thing" but each has small probability.

(iv) In these examples, the expected number of accidents, errors, judicial vacancies is "moderate" ($\mathrm{E}(X) \;=\; \lambda$). Typically λ is between 1 and 20. However, there is no hard upper bound.

18.2 Reminder of facts about the Binomial distribution

(i) From 9.3, 16.2: $P(X=j) \;=\; \binom{n}{j} p^j(1-p)^{n-j}$, for $j=0,1,2,...,n$.

(ii) See 16.2:

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{j=0}^{n} j\binom{n}{j} p^j(1-p)^{n-j} \;=\; \sum_{j=1}^{n} np\binom{n-1}{j-1} p^{j-1}(1-p)^{n-j} \\
&= np\sum_{j=0}^{n-1}\binom{n-1}{j} p^j(1-p)^{n-1-j} \;=\; np. \\
\mathrm{E}(X(X-1)) &= \sum_{j=0}^{n} j(j-1)\binom{n}{j} p^j(1-p)^{n-j} \;=\; \sum_{j=2}^{n} n(n-1)p^2\binom{n-2}{j-2} p^{j-2}(1-p)^{n-j} \\
&= n(n-1)p^2\sum_{j=0}^{n-2}\binom{n-2}{j} p^j(1-p)^{n-2-j} \;=\; n(n-1)p^2. \\
\mathrm{var}(X) &= \mathrm{E}(X(X-1) \;+\; X) - (\mathrm{E}(X))^2 \;=\; n(n-1)p^2 + np - (np)^2 \;=\; np(1-p).
\end{aligned}$$

(iii) Model for number of times something happens in n independent trials (coin tosses, red-flowered offspring pea-plants, Dane speaking German....) when the probability of the "thing" happening on each trial is p

(iv) In these examples, p is "moderate" (0.25, 0.5, ...) and n also usually "moderate" (10 coin tosses, grow 30 pea plants, ...). There is a hard upper bound (n) on the value of X.

18.3 Poisson approximation to the Binomial

(i) Let X be a Binomial ($\mathrm{Bin}(n,p)$) random variable, and Y a Poisson random variable with parameter λ.

(ii) Suppose n gets large, and p gets small in such a way that np remains " moderate". Then we can match up the means: $\mathrm{E}(X) \;=\; np \;=\; \lambda \;=\; \mathrm{E}(Y)$.

(iii) Now $\mathrm{var}(X) \;=\; np(1-p) \;=\; \lambda(1-p) \;\approx\; \lambda \;=\; \mathrm{var}(Y)$.

(iv) In fact,

$$\begin{aligned}
P(X=j) &= \frac{n!}{j!(n-j)!}p^j(1-p)^{n-j} \;=\; \frac{n!}{j!(n-j)!}\left(\frac{\lambda}{n}\right)^j\left(1-\frac{\lambda}{n}\right)^{n-j} \\
&= \frac{n(n-1)...(n-i+1)}{n^i)}\frac{1}{i!}\left(\frac{\lambda}{(1-\lambda/n)}\right)^i(1-\lambda/n)^n \;\approx\; 1\frac{1}{i!}\lambda^i\exp(-\lambda) \;=\; P(Y=j)
\end{aligned}$$

18.4 Back to the class data on birthdays: 37 with 3 pairs

Actual probability of no pairs: $365 \times 364 \times ..... \times 329/(365)^{37} \;=\; 0.1513$.

APPROXIMATION: $n = 37 \times 36/2 = 666$ not-quite-independent pairs, each pair probability $p = 1/365$.

$\lambda = 666/365 = 0.182$, $P(X=0) \;=\; 0.162$.

$P(X \geq 3) \;=\; 1 - P(X=0) - P(X=1) - P(X=2) \;=\; 1 - 0.162 - 0.294 - 0.268 \;=\; 0.275$.