Midterm Exam: Stat 581: Fall 2003
MUE 154: 10.30-11.20, Monday Nov 3, 2003

Attempt all four questions.
You may use any standard theorems/results, but should be clear about which one(s) you are citing.
This is a closed-book, closed-notes exam.

1. State *three* of the following 5 theorems, with enough definition of notation to make your statement clear. For *each of the three that you choose* write one sentence on why it is important and/or useful in the development of Mathematical Statistical theory.

(a) Caratheodory-Hahn Extension theorem

(b) Dominated Convergence Theorem

(c) Theorem of the Unconscious Statistician

(d) Radon-Nikodym theorem

(e) Fubini's Theorem

2. The cdf of a r.v. X with a *Pareto distribution* with parameters (c, k) $(c > 0, k > 0)$ is given by

$$F_X(x) \;=\; P(X \leq x) \;=\; (1 - (k/x)^c) \quad \text{on } x > k.$$

(a) Show that if $k = 1$, the family for varying c is a one-parameter exponential family, and identify the natural parameter and natural statistic.

(b) For varying (c, k), $(c > 0, k > 0)$ is the family an exponential family? Why/why not?

(c) Show that $\log X$ has an exponential distribution on $(\log k, \infty)$.

(d) Hence or otherwise, show that the Pareto family for varying $c > 0$ and $k > 0$ is a group family.

3. Let X_n $n = 1, 2, 3, ...$ be independent random variables defined on a common probability space Ω and such that

$$P(X_n = n^\alpha) \;=\; \frac{1}{n} \quad \text{and} \quad P(X_n = 0) \;=\; 1 - \frac{1}{n} \qquad n = 1, 2, ...$$

where α is a constant. Find the values of α, $-\infty < \alpha < \infty$, for which

(a) X_n converges to 0 in probability,

(b) X_n converges to 0 a.s.,

(c) X_n converges to 0 in r th moment, for given $r > 0$.

(Hint: You may use the fact that for independent events A_n, A_n occurs infinitely often if and only if $\sum_n P(A_n) = \infty$.)

4. Suppose Z_i, $i = 1, ...,$ are i.i.d. Normal $N(0, 1)$. Let $Y_i = Z_i^2$, and $\overline{Y_n} = n^{-1} \sum_{i=1}^{n} Y_i$.

(a) Show that $n^{\frac{1}{2}}(\overline{Y_n} - 1) \to_d N(0, K)$, and find K.

(b) Show that for each $r > 0$, $n^{\frac{1}{2}}((\overline{Y_n})^r - 1) \to_d N(0, V(r))$, and find $V(r)$ as a function of r (and of K if you have not been able to determine K).