Chapter 1: Measure and Random Variables (JAW Ch 0)

1.1 Measurable spaces

(i) Ω **a fixed non-empty set.** $\mathcal{A}$ **non-empty class of subsets.**

(ii) A field: closed under complements and finite unions.

(iii) A σ**-field: closed under complements and countable unions.**

Note: $\cap_1^\infty A_i = (\cup_1^\infty A_i^c)^c$

Examples:

(a) Ω **finite or countable.** 2^Ω **= set of all subsets of** Ω

(b) $\mathcal{C}$ **= some set of subsets of** Ω**,** $\sigma(\mathcal{C})$ **= smallest** σ**-field containing** $\mathcal{C}$**.**

(c) $\Omega = \Re$**:** $\mathcal{C}$ **the set of half-left-open intervals** $(a, b]$ **(including** (c, ∞)**).** $\mathcal{B} = \sigma(C) \equiv$ **Borel sets in** $\Re$**.**

(d) Ω **a metric space, metric** ρ**.** $\mathcal{C}$ **the open sets in** Ω**.** $\mathcal{B} = \sigma(\mathcal{C}) \equiv$ **Borel sets in** Ω**.**

1.2 Measures and probability

(i) Measure $\mu : \mathcal{A} \rightarrow [0, \infty]$, **countably additive, with** $\mu(\Phi) = 0$

(ii) Measure space is triple $(\Omega, \mathcal{A}, \mu)$

(iii) Finite measure: $\mu(\Omega) < \infty$.
Probability Measure: $\mu(\Omega) = 1$
σ**-finite measure:** $\Omega = \cup_1^\infty F_i$, $F_i \in \mathcal{A}$, $\mu(F_i) < \infty$
(For example, Borel sets in $\Re$**.)**

(iv) Caratheodory-Hahn Extension Theorem – **(**σ**-finite) measure** μ **on field** $\mathcal{C}$ **can be extended to (unique** σ**-finite) measure on** $\sigma(\mathcal{C})$**.**

(v) Complete measure space:
$\{B \subset A, \ A \in \mathcal{A}, \ \mu(A) = 0\} \ \Rightarrow \ B \in \mathcal{A} \ (\mu(B) = 0.)$

(vi) Completing spaces:
$\mathcal{A}^* = \{A \cup N; \ A \in \mathcal{A}, \ N \subset B, \ B \in \mathcal{A}, \ \mu(B) = 0\}$
$\mu^*(A \cup N) = \mu(A)$

(vii) Lebesgue-Stieltjes measure: Measure μ **on** $\Re$ **assigning finite values to finite intervals.**

(viii) Generalized df: function F **on** $\Re$**, finite, increasing and right-continuous.**

(xi) Correspondence Theorem: (1-1) correspondence between L-S measure μ **on Borel sets and generalised dfs:** $\mu((a, b]) = F(b) - F(a)$.

(x) For a probability measure P **on** $\Re$**:**
A df F**: increasing, rt. cts., fn on** $\Re$**,** $F(-\infty) = 0, \ F(\infty) = 1$**.**
Correspondence theorem gives (1-1) relationship between P**s and** F**s:** $P((a, b]) = F(b) - F(a)$.

1.3 Measurable functions and random variables (JAW 0.7)

(i) $X: \ \Omega \to \ \Re$ **measurable if** $X^{-1}(B) \in \mathcal{A}$ **for** B **Borel in** $\Re$. **(Note: Sufficient to check** B **of form** (x, ∞)**).**

(ii) $X_1, ..., X_n ...$ **measurable** $\Rightarrow \sup(X_n)$, $\inf(X_n)$, $\overline{\lim}(X_n)$, $\underline{\lim}(X_n)$, $-X_n$ **are measurable (also** $\lim(X_n)$ **if** $\exists$**).**

(iii) Simple function: $X(\omega) = \Sigma_1^m a_i I_{A_i}(\omega)$ **where** A_i **are disjoint,** $A_i \in \mathcal{A}$**,** $a_i \in \Re$**,** $m < \infty$**.**

(iv) Any measurable $X \geq 0$ **is limit of increasing sequence of simple functions.**
Proof: by construction of simple $X_n \nearrow X$**,**
based on $A_0 = I(X > n)$**,** $A_{n,k} = I(X \in ((k-1)2^{-n}, k2^{-n}])$**.**

(v) X **measurable iff it is limit of simple functions.**
Proof: use (ii) and (iv).

(vi) X **and** Y **measurable,** g **measurable** $\Rightarrow X + Y$**,** $X - Y$**,** XY, X/Y, $X^+ = \max(X, 0)$, $X^- = \max(0, -X)$**,** $|X|$ **are measurable. (Proof: use (v))**

(vii) g **a measurable function** $\Re \to \Re$**,** X **measurable** $\Rightarrow$ $g(X)$ **measurable since** $(gX)^{-1}(B) = X^{-1}(g^{-1}(B))$**.**
g **continuous** $\Rightarrow g$ **measurable, since** g^{-1}**(open) is open.**

1.4 Integration and integrability (JAW 0.8-9)

(i) For X **simple,** $X = \Sigma_1^m a_i I_{A_i}$**, define** $\int X d\mu \; = \; \Sigma_1^m a_i \mu(A_i)$**.**
For $X \geq 0$**,** X **measurable,** $\exists\ X_n$ **simple,** $X_n \nearrow X$**, define** $\int X d\mu = \lim_n (\int X_n d\mu)$**.**
For $X = X^+ - X^-$**, define** $\int X d\mu \; = \; \int X^+ d\mu - \int X^- d\mu$ **provided at least one of these** $< \infty$**.**
If $|\int X d\mu| < \infty$**,** X **is integrable. (Equiv.** $\int |X| d\mu < \infty$**)**

(ii) Then for integrable $X,\ Y$**,** $\int (X+Y) d\mu \; = \; \int X d\mu + \int Y d\mu$**,** $X \geq Y \;\; \Rightarrow \;\; \int X d\mu \geq \int Y d\mu$**, etc.**

(iii) Three important theorems:
MCT: $0 \leq X_n \nearrow X$**, then** $\int X_n d\mu \to \int X d\mu$**.**
Fatou's lemma: $0 \leq X_n$**, then** $\int \liminf X_n d\mu \;\leq\; \liminf(\int X_n d\mu)$**.**
DCT: $|X_n| \leq Y$**,** Y **integrable, and** $X_n \to X$ **except perhaps on a set** N **with** $\mu(N) = 0$**, then** $\int |X_n - X| d\mu \to 0$ **and** $\lim \int X_n d\mu \; = \; \int X d\mu$**.**

(iv) Let $(\Omega, \mathcal{A}, P)$ **be a probability space.**
A real-valued random variable X **is a finite measurable function** $\Omega \to \Re$**. For** B **a Borel set in** $\Re$**,**
$P_X(B) \;\equiv\; P(X \in B) \;=\; P(\{\omega \in \Omega : X(\omega) \in B\})$**. The asociated df is** $F_X(-\infty) = 0,\ F_X(\infty) = 1, F_X(x) = P_X((-\infty, x])$**,**
Then $(\Re, \mathcal{B}, P_X)$ **is a probability space.**
Theorem of the unconscious statistician:
$\int_\Omega g(X(\omega)) dP \;=\; \int_\Re g(x) dP_X \;\equiv\; \int_\Re g(x) dF_X(x)$**.**

1.5 Absolute continuity and densities. JAW 0.11

(i) $(\Omega, \mathcal{A}, \mu)$ **a measure space.** $X : \Omega \to \Re$ **measurable fn.,** $X \geq 0$. **Define** $\nu(A) = \int_A X d\mu \equiv \int_\Omega X I_A d\mu$.
Then ν **is also a measure, and is finite iff** X **is integrable.**

(ii) We say, ν **has density** X **w.r.t.** μ

(iii) $\mu(A) = 0 \Rightarrow \nu(A) = 0$: $\nu \ll \mu$ **(**ν **is absolutely continuous w.r.t.** μ, **or** μ **dominates** ν**).**

(iv) Radon-Nikodym theorem
$(\Omega, \mathcal{A}, \mu)$ **a measure space.** μ σ**-finite.** $\nu \ll \mu$.
Then $\exists$ **measurable** $X \geq 0$ **s.t.** $\nu(A) = \int_A X d\mu$ **for all** $A \in \mathcal{A}$.
Further, X **is unique a.e.(**μ**).**
Write $X = \frac{d\nu}{d\mu} \equiv$ **Radon-Nikodym derivative of** ν **w.r.t.** μ.

(v) Change of variable theorem
$(\Omega, \mathcal{A}, \mu)$, $\nu \ll \mu$ **as above.** Z **measurable and** $\int Z d\nu$ **well defined. Then, for all** $A \in \mathcal{A}$, $\int_A Z d\nu = \int_A Z \frac{d\nu}{d\mu} d\mu$.
Proof: (see JAW 0.11-12):
$Z = I_B$, Z **simple,** $Z \geq 0$, $Z = Z^+ - Z^-$.

(vi) Example: real-valued vector random variables:
Probability space is $(\Re^n, \mathcal{B}_n, P)$.
We assume P **has a density** f **w.r.t.** σ**-finite measure** μ.
If μ **is Lebesgue measure of** $\Re^n$**: f is pdf.**
If μ **is counting measure on countable set,** f **is pmf.**

1.6 Product spaces and product measures. (JAW 0.13)

(i) Product measures

Let $(X, \mathcal{X}, \mu)$ and $(Y, \mathcal{Y}, \nu)$ be σ-finite measure spaces.
Let $A \in \mathcal{X}$, $B \in \mathcal{Y}$: $A \times B \equiv \{(x, y); x \in A, y \in B\}$.
Define $\mathcal{X} \times \mathcal{Y} \equiv \sigma(\{A \times B : A \in \mathcal{X}, B \in \mathcal{Y}\}$
Let $\pi(A \times B) = \mu(A)\nu(B)$ for "measurable rectangle" $A \times B$.
π is "product measure" on $\mathcal{X} \times \mathcal{Y}$.

(ii) Fubini's Theorem

Suppose $f : \mathcal{X} \times \mathcal{Y} \rightarrow [0, \infty)$ is $\mathcal{X} \times \mathcal{Y}$ measurable. Then $\int_{\mathcal{Y}} f(x, y) d\nu(y)$ is $\mathcal{X}$-measurable.
and $\int_{\mathcal{X}} f(x, y) d\mu(y)$ is $\mathcal{Y}$-measurable.

$$\begin{aligned} \int_{\mathcal{X} \times \mathcal{Y}} f(x, y) d\pi(x, y) &= \int_{\mathcal{X}} \left(\int_{\mathcal{Y}} f(x, y) d\nu(y) \right) d\mu(x) \\ &= \int_{\mathcal{Y}} \left(\int_{\mathcal{X}} f(x, y) d\mu(y) \right) d\nu(y). \end{aligned}$$

If f is s.t. $\int |f| d\pi < \infty$, then above is true even if f is not ≥ 0.