Midterm Exam: Stat 581: Fall 2004
SMI 405: 10.30-11.20, Nov 3

Attempt all four questions. Justify your answers (briefly).
You may use any standard theorems/results, but should be clear about which one(s) you are citing.
This is a closed-book, closed-notes exam.

1. Let U be uniformly distributed $U(0, 1)$.
Consider the collection of random variables $X_\alpha \equiv U^\alpha$, $\alpha > 0$.

(a) Show that this defines a group family, and determine the density of X_α.

(b) Show that this defines an exponential family. What are the natural parameter and natural statistic?

2. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independently distributed according to $N(\xi, \sigma^2)$ and $N(\eta, \tau^2)$ respectively. Find the minimal sufficient statistics for the cases (a), (b) and (c) below, and, in each case, say whether you expect the minimal sufficient statistic to be complete, and why. (No proofs required.)

(a) $\xi,\ \eta,\ \sigma,\ \tau$ are arbitrary: $-\infty < \xi, \eta < \infty$, $0 < \sigma, \tau$.

(b) $\sigma = \tau$,with $\xi,\ \eta,\ \sigma$ arbitrary ($\sigma > 0$).

(c) $\xi = \eta$, with $\xi,\ \sigma,\ \tau$ arbitrary ($\sigma, \tau > 0$).

3. Let the probability space be $(\Omega, \mathcal{B}, \lambda)$, where $\Omega = [0, 1]$, $\mathcal{B}$ is the Borel sets in Ω, and λ is Lebesgue measure. Let Z be a random variable on the probability space and suppose Z is uniformly distributed $U(0, 1)$.

(a)Let $X_n = I_{[m2^{-k},(m+1)2^{-k})}(Z)$, where $n = 2^k + m$, and $0 \leq m < 2^k$.
Does $X_n \to_{a.s.} 0$?
Does $X_n \to_r 0$, for any $r > 0$ (i.e. converge in rth moment)?
Does $X_n \to_p 0$?

(b) Let $Y_n \ = \ 2^n I_{(0,1/n)}(Z)$.
Does $Y_n \to_{a.s.} 0$?
Does $Y_n \to_r 0$, for any $r > 0$?
Does $Y_n \to_p 0$?

4. Let $X_1, \ldots, X_n$ be independent Bernoulli random variables:

$$\Pr(X_i = 1) \ = \ p, \quad \Pr(X_i = 0) \ = \ (1 - p) : \quad \operatorname{var}(X_i) \ = \ p(1 - p).$$

Let $\overline{X_n} \ = \ n^{-1} \sum_1^n X_i$. Consider $\sqrt{\overline{X_n}(1 - \overline{X_n})}$ as an estimator of the Bernoulli standard deviation $\sqrt{p(1 - p)}$. Assuming $p \neq 1/2$, what is the standardized limiting distribution of this estimator?