Stat 581 Homework 8: Outline solutions

1. X_i i.i.id, $\mathrm{E}(X_i) = \theta$, $\mathrm{var}(X_i) = 1$, $m_4 = \mathrm{E}(X_i^4) < \infty$, $\mu_4 = \mathrm{E}(X_i - \theta)^4$
$T_{1,n} = n^{-1}\sum_{i=1}^n X_i^2 - 1$, $T_{2,n} = (n^{-1}\sum_{i=1}^n X_i)^2 \; - \; (1/n)$

(a) $\mathrm{E}(T_{1,n}) \; = \; \mathrm{E}(X_i^2) - 1 \; = \; 1 + \theta^2 - 1 \; = \; \theta^2$. $\mathrm{E}(T_{2,n}) \; = \; \theta^2 + (1/n) - (1/n) = \theta^2$.
$\sqrt{n}(T_{2,n} - \theta^2) \; = \; \sqrt{n}(\overline{X_n}^2 - \theta^2 - n^{-1}) \; = \; \sqrt{n}(\overline{X_n} - \theta)(\overline{X_n} + \theta) - n^{-1/2} \;\; \to_d 2\theta N(0,1) \equiv N(0, 4\theta^2)$
by cts mapping thm, since $\overline{X_n} \to_{a.s} \theta$, $n^{-1/2} \to 0$.
$\sqrt{n}(T_{1,n} - \theta^2) \; = \; \sqrt{n}(\overline{X_i^2} - (1+\theta^2)) \; \to_d \; N(0, \mathrm{var}(X_i^2))$, where $\mathrm{var}(X_i^2) \; = \; m_4 - (1+\theta^2)^2$.

(b) If dsn is symmetric, $0 \; = \; \mathrm{E}((X-\theta)^3) \; = \; m_3 - 3\theta(\theta^2+1) + 2\theta^3$; $m_3 \; = \; \theta^3 + 3\theta$.
So $\mu_4 \; = \; m_4 - 4\theta m_3 + 6\theta^2(1+\theta^2) - 4\theta^4 + \theta^4 \; = \; m_4 - 6\theta^2 + \theta^4$.
Now ARE of $(T_{2,n})$ to $(T_{1,n})$ is $(m_4 - (1+\theta^2)^2)/(4\theta^2) \; = \; 1 + (\mu_4 - 1)/(4\theta^2)$.
But $\mu_4 \; = \; \mathrm{E}((X_i - \theta)^4) \; \geq \; (\mathrm{E}((X_i-\theta)^2))^2 \; = \; 1$, so $e_{2,1} \; \geq \; 1$.

(c) Many choices will do – need a dsn skewed to left: $\theta < 0$ and/or $\mu_3 < 0$.
e.g. $B(k,p)$ for large p, $kp = \theta$, with (e.g) $p = 0.95$, $kp = 20$
or $X \sim \mathcal{E}(1) + \theta - 1$ works for $\theta < -1$. etc.

2. $X(u) \sim N(\alpha + \beta u, \sigma^2)$, $0 \leq u \leq 1$, σ^2 known, n even. $\tilde{\beta} \; = \; \sum_i (u_i - \overline{u}) X_i \; / \; \sum_i (u_i - \overline{u})^2$
(a)

$$\begin{aligned}
\ell_n &= \text{const} - \sum_{i=1}^n (x_i - (\alpha + \beta u_i))^2/(2\sigma^2) \\
\frac{\partial \ell_n}{\partial \alpha} &= \frac{n\overline{x} - n\alpha - \beta\sum_i u_i}{\sigma^2}, \quad \frac{\partial \ell_n}{\partial \beta} \; = \; \frac{\sum_i x_i u_i - \alpha \sum_i u_i \; - \beta \sum_i u_i^2}{\sigma^2} \\
-\frac{\partial^2 \ell_n}{\partial \alpha^2} &= \frac{n}{\sigma^2}, \quad -\frac{\partial^2 \ell_n}{\partial \alpha \partial \beta} \; = \; \frac{n\overline{u}}{\sigma^2}, \quad -\frac{\partial^2 \ell_n}{\partial \beta^2} \; = \; \frac{\sum_i u_i^2}{\sigma^2} \\
\det(I(\alpha,\beta)) &= \left(\frac{n}{\sigma^2}\right)^2 (\overline{u^2} \; - \; \overline{u}^2) \; = \; \frac{n}{\sigma^4}\sum_i (u_i - \overline{u})^2 \\
\mathrm{var}(\tilde{\beta}) &= \sum_i (u_i - \overline{u})^2 \sigma^2 / (\sum_i (u_i - \overline{u})^2)^2 \; = \; \sigma^2 / \sum_i (u_i - \overline{u})^2
\end{aligned}$$

(b) Now $\sum_i (u_i - \overline{u})^2$ is max by having all u_i either 0 or 1, and then distance from $\overline{u}$ is max by putting $n/2$ at each. So $n_0 = n_1 = n/2$ maximizes $\det(I(\alpha,\beta))$ and minimizes $\mathrm{var}(\tilde{\beta})$.

(c) $\mathbf{u} \; = \; (0,1,2,3,,,,,,(n-1))/(n-1)$: n points equally spaced. $\overline{u} = 1/2$.

$$\sum_{i=1}^n (u_i - \overline{u})^2 \; = \; \sum_{i=1}^n \left(\frac{i-1}{n-1} - \frac{1}{2}\right)^2 \; = \; \frac{n(n+1)}{12(n-1)}$$

to be compared with $n/4$ so A.R.E is $(n+1)/3(n-1) \; \to \; 1/3$.
(Note $\sum_i (u_i - \overline{u})^2 \; \propto 1/\mathrm{var}$ and both estimators have var $\sim 1/n$.)

(d) (b) is more efficient, but (c) is more robust.

3. $\ell_n \; = \; -\lambda n + S_x \log\lambda \; - \mu n + S_y \log\mu$
(a) Unconstrained MLE is $\widehat{\lambda} = \overline{X_n}$, $\widehat{\mu} = \overline{Y_n}$. If $\lambda = 2\mu$, $\tilde{\lambda} = 2\tilde{\mu} = 2(S_x + S_y)/3n$

$$\begin{aligned}
2(\ell_n(\widehat{\lambda}, \widehat{\mu}) \; - \; \ell_n(\tilde{\lambda}, \tfrac{1}{2}\tilde{\lambda})) &= 2(\overline{X_n}\log\overline{X_n} \; + \; \overline{Y_n}\log\overline{Y_n} \\
&\quad - (\overline{X_n} \; + \; \overline{Y_n})\log(\overline{X_n} + \overline{Y_n}) - \overline{X_n}\log(2/3) - \overline{Y_n}\log(1/3)) \\
&= 2\left(\overline{X_n}\log\left(\frac{3\overline{X_n}}{2(\overline{X_n} + \overline{Y_n})}\right) \; + \; \overline{Y_n}\log\left(\frac{3\overline{Y_n}}{(\overline{X_n} + \overline{Y_n})}\right)\right)
\end{aligned}$$

and statistic is χ^2_1 if null hypothesis is true.

(b) $(\bigtriangledown \ell_n)^t = n((\overline{X_n}/\lambda) - 1, (\overline{Y_n}/\mu) - 1)$,
$n^{-1}J_n = \mathrm{diag}(\overline{X_n}/\lambda^2, \overline{Y_n}/\mu^2)$, $I^{-1} = (\mathrm{E}(n^{-1}J_n))^{-1} = \mathrm{diag}(\lambda, \mu)$.
Rao statistic is evaluated at constrained MLE $\tilde{\lambda} = 2\tilde{\mu} = 2(\overline{X_n} + \overline{Y_n})/3$.
$R_n = n(\overline{X_n} - 2\overline{Y_n})^2/(2(\overline{X_n} + \overline{Y_n}))$, which makes sense as under the null hypothesis $(\overline{X_n} - 2\overline{Y_n})$ has mean 0 and variance $3\lambda/n$, and $2(\overline{X_n} + \overline{Y_n}) \to_p 3\lambda$.

4 $W_n = n\widehat{\beta}^2 I_{\beta\beta\cdot\mu}$
(a) $\beta = \lambda - 2\mu$.

$$
\begin{aligned}
I(\lambda, \mu) &= \begin{pmatrix} 1/\lambda, & 0 \\ 0. & 1/\mu \end{pmatrix}, \\
I(\beta, \mu) &= \left(\frac{\partial(\lambda,\mu)}{\partial(\beta,\mu)}\right)^t I(\lambda,\mu) \left(\frac{\partial(\lambda,\mu)}{\partial(\beta,\mu)}\right) \\
\left(\frac{\partial(\lambda,\mu)}{\partial(\beta,\mu)}\right) &= \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\
I(\beta, \mu) &= \begin{pmatrix} 1/\lambda & 2/\lambda \\ 2/\lambda & (4/\lambda) + (1/\mu) \end{pmatrix}, \qquad I_{\beta\beta\cdot\mu} = 1/(4\mu + \lambda) \\
W_n &= n(\overline{X_n} - 2\overline{Y_n})^2/(4\overline{Y_n} + \overline{X_n})
\end{aligned}
$$

Note $\mathrm{var}(\overline{X_n} - 2\overline{Y_n}) = \lambda + 4\mu$, $4\overline{Y_n} + \overline{X_n} \to 4\mu + \lambda$.

(b) $\beta = \mu/\lambda$

$$
\begin{aligned}
\left(\frac{\partial(\lambda,\mu)}{\partial(\beta,\mu)}\right) &= \begin{pmatrix} -\mu/\lambda^2 & 1/\lambda \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -\lambda^2/\mu & \lambda/\mu \\ 0 & 1 \end{pmatrix} \\
I(\beta, \mu) &= \begin{pmatrix} \lambda^3/\mu^2 & -\lambda^2/\mu^2 \\ -\lambda^2/\mu^2 & (\lambda/\mu^2) + (1/\mu) \end{pmatrix}, I_{\beta\beta\cdot\mu} = \lambda^3/(\mu(\lambda + \mu)) \\
W_n &= n(\overline{X_n} - 2\overline{Y_n})^2 \overline{X_n}/(4\overline{Y_n}(\overline{Y_n} + \overline{X_n}))
\end{aligned}
$$

Note $(4\overline{Y_n}(\overline{Y_n} + \overline{X_n}))/\overline{X_n} \to_p 6\mu$ if $\lambda = 2\mu$.

(c) $\beta = \lambda/\mu$

$$
\begin{aligned}
\left(\frac{\partial(\lambda,\mu)}{\partial(\beta,\mu)}\right) &= \begin{pmatrix} 1/\mu & -\lambda/\mu^2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \mu & \lambda/\mu \\ 0 & 1 \end{pmatrix} \\
I(\beta, \mu) &= \begin{pmatrix} \mu^2/\lambda & 1 \\ 1 & (\lambda/\mu^2) + (1/\mu) \end{pmatrix}, I_{\beta\beta\cdot\mu} = \mu^3/(\lambda(\lambda + \mu)) \\
W_n &= n(\overline{X_n} - 2\overline{Y_n})^2 \overline{Y_n}/(\overline{X_n}(\overline{Y_n} + \overline{X_n}))
\end{aligned}
$$

Note $(\overline{X_n}(\overline{Y_n} + \overline{X_n}))/\overline{Y_n} \to_p 6\mu$ if $\lambda = 2\mu$.

(d) $\beta = \log(\mu/\lambda)$

$$
\begin{aligned}
\left(\frac{\partial(\lambda,\mu)}{\partial(\beta,\mu)}\right) &= \begin{pmatrix} -1/\lambda & 1/\mu \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -\lambda & \lambda/\mu \\ 0 & 1 \end{pmatrix} \\
I(\beta, \mu) &= \begin{pmatrix} \lambda & -\lambda/\mu \\ -\lambda/\mu & (\lambda/\mu^2) + (1/\mu) \end{pmatrix}, I_{\beta\beta\cdot\mu} = \lambda\mu/(\lambda + \mu) \\
W_n &= n(\log(2\overline{Y_n}/\overline{X_n}))^2 \overline{Y_n X_n}/(\overline{Y_n} + \overline{X_n})
\end{aligned}
$$

Note expanding the log of $(1+(2\overline{Y_n}/\overline{X_n}\ -1))$ to first order gives (c) again, while expanding negative log of $(1+((\overline{X_n}/2\overline{Y_n})-1))$ gives (b).

5. $Nij \sim Mn_4(n, p_{ij})$, $i, j = 1, 2$
(a) Unconstrained $\widehat{p_{ij}} = N_{ij}/n$, $\widehat{p_{1\cdot}} = N_{1\cdot}/n$, $\widehat{p_{\cdot 1}} = N_{\cdot 1}/n$,
$\widehat{\psi} = \log(\widehat{p_{12}p_{21}}/\widehat{p_{11}p_{22}}) \ = \ \log(N_{12}N_{21}/N_{11}N_{22})$.
(b)

$$\begin{aligned}
\psi &= g(\mathbf{p}) = \log p_{12} + \log p_{21} - \log p_{11} - \log p_{22} \\
\bigtriangledown g(\mathbf{p}) &= (-1/p_{11}, 1/p_{12}, 1/p_{12}, -1/p_{22}) \\
\sqrt{n}(\widehat{\mathbf{p}} - \mathbf{p}) &\to_d Z \sim N_4(0, V), \text{ where } V \ = \ \text{diag}\mathbf{p} - \mathbf{p}\mathbf{p}' \\
\text{So } \sqrt{n}(\widehat{\psi} - \psi) &\to_d \bigtriangledown g(p) Z \sim N(0, v) \text{ where} \\
v &= (\bigtriangledown g)^t V (\bigtriangledown g) \ = \ p_{11}^{-1} + p_{12}^{-1} + p_{21}^{-1} + p_{22}^{-1}
\end{aligned}$$

(c) $H_0:\ p_{ij} = p_{i\cdot}p_{\cdot j}$, or $\psi = 0$. If $\psi = 0$, $\sqrt{n}(\exp(\widehat{\psi}) - \exp(0)) \to_d N(0, v)$, since $e^0 = 1$.

$$\begin{aligned}
Q_n &= n\sum_i\sum_j (N_{ij} - N_{i\cdot}N_{\cdot j})^2/N_{i\cdot}N_{\cdot j} \ = \ n^{-3}(N_{11}N_{22} - N_{12}N_{21})^2 \sum_i\sum_j (\widehat{p_{i\cdot}}\widehat{p_{\cdot j}})^{-1} \\
&= \frac{n(\exp(\widehat{\psi_n}) - 1)^2 \widehat{p_{11}}^2 \widehat{p_{22}}^2}{\widehat{p_{1\cdot}}\widehat{p_{2\cdot}}\widehat{p_{\cdot 1}}\widehat{p_{\cdot 2}}} \ \to_d \ (N(0, v))^2 p_{1\cdot}p_{2\cdot}p_{\cdot 1}p_{\cdot 2} \\
&\equiv (N(0,1))^2 v^{-1} p_{1\cdot}p_{2\cdot}p_{\cdot 1}p_{\cdot 2} \ \equiv \ (N(0,1))^2 \ \sim \ \chi_1^2 \text{ if } H_0 \text{ true.}
\end{aligned}$$

(d)

$$n^{-1}Q_n \ \equiv \ \frac{(\widehat{p_{12}p_{21}} - \widehat{p_{11}p_{22}})^2}{\widehat{p_{1\cdot}}\widehat{p_{2\cdot}}\widehat{p_{\cdot 1}}\widehat{p_{\cdot 2}}} \ \to \ \frac{(p_{12}p_{21} - p_{11}p_{22})^2}{p_{1\cdot}p_{2\cdot}p_{\cdot 1}p_{\cdot 2}}$$

(e) Under local alternatives $\sqrt{n}\widehat{\psi_n} \to_d N(t, v)$ so $Q_n \to_d\ (N(t\sqrt{c}, 1))^2 \ =_d\ \chi_1^2(\delta)$ where $\delta = ct^2$, and $c = p_{1\cdot}p_{2\cdot}p_{\cdot 1}p_{\cdot 2}$.
$\mathbf{p}^{(0)} = (0.3, 0.2, 0.1, 0.4)$; $c = 0.06$, $tn^{-1/2} \ = \ \psi \ = \ \log(p_{12}p_{21}/p_{11}p_{22})$, $t = -9.814$.
So $\delta = (9.814)^2(0.06) = 5.8$, and power is

$$\begin{aligned}
P(\chi_1^2(5.8) \ > \ [\chi_1^2]^{0.95} = 3.84) &= P(N(2.4, 1) > 1.96) + P(N(2.4, 1) < -1.96) \\
&\approx P(N(0, 1) > -0.5) \ \approx \ 2/3.
\end{aligned}$$

Oops: this year $\alpha = 0.02$, so power is

$$\begin{aligned}
P(\chi_1^2(5.8) \ > \ [\chi_1^2]^{[0.98]} = 5.412) &= P(N(2.4, 1) > 2.3) + P(N(2.4, 1) < -2.3) \\
&\approx P(N(0, 1) > -0.1) \ \approx \ 1/2.
\end{aligned}$$

(Actually, 0.53, according to those who did the computation more accurately.)