Stat 581,2004: Homework 6: Outline solutions

1. (a) $X_1, \ldots, X_n$ i.i.d $\sim\ f(x;\theta)$. Using indep and i.d.:

$$\mathrm{E}_{\theta_0}\left(\left(\frac{f(\mathbf{x};\theta)}{f(\mathbf{x};\theta_0)}\right)^2\right) \;=\; \prod_1^n\left(\mathrm{E}_{\theta_0}\left(\frac{f(x_i;\theta)}{f(x_i;\theta_0)}\right)^2\right) \;=\; \left(\mathrm{E}_{\theta_0}\left(\frac{f(x_1;\theta)}{f(x_1;\theta_0)}\right)^2\right)^n \;=\; H^n \text{ say.}$$

Also

$$\mathrm{E}_{\theta_0}\left(\frac{f(\mathbf{x};\theta)}{f(\mathbf{x};\theta_0)}\right) \;=\; 1 \text{ so } \mathrm{E}_{\theta_0}\left(\left(\frac{f(\mathbf{x};\theta)}{f(\mathbf{x};\theta_0)} \;-\; 1\right)^2\right) \;=\; H^n - 2 + 1 \;=\; H^n - 1$$

and Hammersley-Chapman-Robbins gives $\mathrm{var}_\theta(T(X^{(n)})) \;\geq\; \Delta^2/(H^n - 1)$ where $\mathrm{E}_\theta(T(X^{(n)})) = \theta$ and $\Delta = \theta - \theta_0$.

(b) Evaluating H in this case

$$\begin{aligned}
H \;&=\; \mathrm{E}(\exp(-\frac{1}{\sigma^2}(-2X(\theta-\theta_0)+\theta^2-\theta_0^2))) \text{ where } X \sim N(\theta_0,\sigma^2)\\
&=\; \mathrm{E}(\exp(2(X-\theta_0)\Delta/\sigma^2))\exp(\Delta(2\theta_0-(\theta+\theta_0))/\sigma^2)\\
&=\; \exp(\frac{1}{2}(2\Delta/\sigma^2)^2\sigma^2)\exp(-\Delta^2/\sigma^2) \;=\; \exp(\Delta^2/\sigma^2) \quad \text{or}\\
\mathrm{var}(T) \;&\geq\; \Delta^2/(e^{n\Delta^2/\sigma^2} \;-\; 1) \quad \text{or}\\
\mathrm{var}(T) \;&\geq\; \frac{\sigma^2}{n}\sup_x(x/(e^x-1)), \quad \text{where } \; x \;=\; \frac{n}{\sigma^2}(1,4,9,16,\ldots.)
\end{aligned}$$

But $x/(e^x - 1) \;=\; (1 + (x/2) + (x^2/6) + \ldots)^{-1}$ which is a decreasing function of x, so sup is at $x = n/\sigma^2$ and $\mathrm{var}(T) \;\geq\; 1/(\exp(n/\sigma^2) - 1)$.

2. (a) $\log L \;=\; n\log\sigma - (S^2 + n(\overline{X_n} - \theta)^2)/2\sigma^2$ which is monotone in $(\overline{X_n} - \theta)^2$ or $|\overline{X_n} - \theta|$, so MLE $\widehat{\theta_n}$ is integer closest to $\overline{X_n}$ or or $\widehat{\theta_n} \;=\; \lfloor \overline{X_n} + 0.5 \rfloor$.
$\overline{X_n} \sim N(\theta, \sigma^2/n)$, which is symmetric about θ, so dsn of $\widehat{\theta_n}$ is symmetric about θ, so $\mathrm{E}(\widehat{\theta_n}) \;=\; \theta$.

(b) $\widehat{\theta_n} \;=\; \theta$ if $|\overline{X_n} - \theta| \;<\; 1/2$ or $\sqrt{n}/2\sigma \;>\; |\overline{X_n} - \theta|\sqrt{n}/\sigma \;\sim\; N(0,1)$. So $\Pr(\widehat{\theta_n} = \theta) \;=\; \Phi(\sqrt{n}/2\sigma) - \Phi(-\sqrt{n}/2\sigma) = 0.95$ if $\sqrt{n}/2\sigma = 1.96$, or $n = (3.92)^2\sigma^2$, so $n \geq 15.5\sigma^2$ will do.

(c) $\Pr(\widehat{\theta_n} = \theta) \;=\; 2\Phi(\sqrt{n}/2\sigma) \;-\; 1 \;\;\rightarrow\;\; 1$, so consistent.

(d) $\Pr(\widehat{\theta_n} = \theta + k) \;=\; \Phi((2k+1)\sqrt{n}/2\sigma) \;-\; \Phi((2k-1)\sqrt{n}/2\sigma)$ (for $k \geq 0$ and symmetric for negative k). So

$$\begin{aligned}
\mathrm{var}(\widehat{\theta_n}) \;&=\; \mathrm{E}((\widehat{\theta_n} - \theta)^2) \;=\; 2\sum_{k=1}^{\infty} k^2(\Phi((k+\frac{1}{2})\sqrt{n}/\sigma) \;-\; \Phi((k-\frac{1}{2})\sqrt{n}/\sigma))\\
&=\; 2\sum_{k=1}^{\infty} k^2((1 \;-\; \Phi((k-\frac{1}{2})\sqrt{n}/\sigma)) \;-\; (1 - \Phi((k+\frac{1}{2})\sqrt{n}/\sigma)))\\
&=\; 2\sum_{k=1}^{\infty}(1 \;-\; \Phi((k-\frac{1}{2})\sqrt{n}/\sigma))(k^2 - (k-1)^2)\\
&=\; 2\sum_{k=1}^{\infty}(2k-1)(1 \;-\; \Phi((k-\frac{1}{2})\sqrt{n}/\sigma))
\end{aligned}$$

3. $\log f = -\frac{1}{2}\log(1-\rho^2) - (x^2 - 2\rho xy + y^2)/(2(1-\rho^2))$. Let $\alpha = (1-\rho^2)^{-1}$, $\rho = g(\alpha) = ((\alpha-1)/\alpha)^{\frac{1}{2}}$. Then

$$\begin{aligned}
\log f &= \frac{1}{2}\log\alpha - \frac{1}{2}\alpha(x^2 - 2xyg(\alpha) + y^2) \\
\frac{\partial \log f}{\partial \alpha} &= (2\alpha)^{-1} \; + \; xyg(\alpha) \; + \; \alpha xyg'(\alpha) \; + \; \text{const} \\
\frac{\partial^2 \log f}{\partial \alpha^2} &= -(2\alpha^2)^{-1} \; + \; 2xyg'(\alpha) + \alpha xyg''(\alpha) \\
g'(\alpha) &= \frac{1}{2}((\alpha-1)/\alpha)^{-\frac{1}{2}}/\alpha^2 \quad = \quad (2\rho\alpha^2)^{-1} \\
\log g'(\alpha) &= -\log 2 \; - \; \log(g(\alpha)) - 2\log(\alpha) \\
\frac{g''(\alpha)}{g'(\alpha)} &= -\frac{g\text{''}(\alpha)}{g(\alpha)} - 2/\alpha \quad = \quad -(2\rho^2\alpha^2)^{-1} - 2/\alpha. \\
I(\alpha) &= -\mathrm{E}\left(\frac{\partial^2 \log f}{\partial \alpha^2}\right) = (2\alpha^2)^{-1} \; - \; 2\rho g'(\alpha) - \alpha\rho g''(\alpha) \quad = \quad (2\alpha-1)/(4\alpha^2(\alpha-1)) \\
I(\rho) &= \left(\frac{\partial\alpha}{\partial\rho}\right)^2 I(\alpha) \quad = \quad 4\rho^2\alpha^4 I(\alpha) \quad = \quad \alpha(2\alpha-1) \; = \; (1+\rho^2)/(1-\rho^2)^2
\end{aligned}$$

So CRLB is $(1-\rho^2)^2/(1+\rho^2)$, but from a prev. hwk we know $\sqrt{n}(r-\rho) \;\; \to_d \;\; N(0,(1-\rho^2)^2)$, so r does not achieve the lower bound asymptotically. There is no reason why is should – it is not the MLE.

4. (a) $\log f \; = \; (\alpha-1)\log y \; + \; \alpha\log\beta \; - \; \beta y \; - \; \log\Gamma(\alpha)$.
Differentiating twice and setting $\Psi(\alpha) \; = \; \frac{\partial}{\partial\alpha}(\Gamma'(\alpha)/\Gamma(\alpha))$, gives immediately

$$I(\alpha,\beta) \quad = \quad \begin{pmatrix} \Psi(\alpha), & -1/\beta \\ -1/\beta, & \alpha/\beta^2 \end{pmatrix}$$

(b) Now to find ϕ orthogonal to α we set

$$\frac{\partial\beta}{\partial\alpha} \quad = \quad -(\beta^2/\alpha)(-1/\beta) \quad = \quad \beta/\alpha$$

giving $\log\beta \; = \; \log\alpha \; + \; C(\phi)$ or $\phi = \beta/\alpha$.
This one is easily checked: substituting $\beta = \alpha\phi$ in $\log f$ we find

$$\frac{\partial^2 \log f}{\partial\alpha\partial\phi} \quad = \quad 1/\phi \; - \; y \;\; \text{but } \mathrm{E}(Y) \; = \; \alpha/\beta \; = \; 1/\phi.$$

To find η orthogonal to β we set

$$\frac{\partial\alpha}{\partial\beta} \quad = \quad -(\Psi(\alpha))^{-1}(-1/\beta) \quad = \quad 1/(\beta\Psi(\alpha))$$

Integrating $\Gamma'(\alpha)/\Gamma(\alpha) \; = \; \log\beta + C(\eta)$ giving $\eta \; = \; \beta^{-1}\exp(\Gamma'(\alpha)/\Gamma(\alpha))$.

5. (a) If $\theta \neq 0$, $\Pr(\overline{X_n} < n^{-1/4}) \to 0$, so $\sqrt{n}(T_n - \overline{X_n}) \to_p \ 0$, $\sqrt{n}(T_n - \theta) \to_d \ N(0,1)$
If $\theta = 0$, $\Pr(\overline{X_n} < n^{-1/4}) \to 1$, so $\sqrt{n}(T_n - a\overline{X_n}) \to_p \ 0$, $\sqrt{n}T_n \to_d \ N(0, a^2)$
This example shows that regularity of f_θ is not enough – see JAW notes 3.25.

(b)

$$\begin{aligned}
f_\theta(x) &= \frac{1}{2}(\sqrt{2\pi})^{-1}(\exp(-\frac{1}{2}(x-\theta)^2) \ + \ \exp(-\frac{1}{2}(x+\theta)^2)) \\
\log f_\theta(x) &= \text{const} \ - \ \frac{1}{2}\theta^2 \ + \ \log(e^{x\theta} + e^{-x\theta}) \\
\frac{\partial^2 \log f}{\partial \theta^2} &= -1 \ + \ \frac{4x^2 \exp(2x\theta)}{(1+\exp(2x\theta))^2} \\
I_n(\theta) &= nI_1(\theta) \ = \ n\mathrm{E}_\theta \left(1 \ - \ \frac{4x^2 \exp(2x\theta)}{(1+\exp(2x\theta))^2}\right) \\
I_n(0) &= n\mathrm{E}(1 - X^2) \ = \ 0 \text{ as } f_0(x) \equiv N(0,1)
\end{aligned}$$

So for any unbiased estimator T_n of θ, $\mathrm{var}_\theta(T_n) \to \infty$ as $\theta \to 0$.
Estimating θ only makes sense when θ is clearly non-zero.
One could first test $H_0: \ \theta = 0$, and estimate θ only if this is rejected. For example, one could do a LR test.