Stat 581 Homework 4: Outline solutions

1. (a) $\Pr(X_n \neq 0) \;=\; 1/n \;\to\; 0$, so $X_n \to_p 0$ for all real α.

(b) From the Ferguson example, $X_n \;=\; n^\alpha$ i.o. So if $\alpha \geq 0$, X_n does not cge a.s. to 0. However, if $\alpha < 0$, $n^\alpha \to 0$. So $X_n \to_{a.s.} 0$, iff $-\infty < \alpha < 0$.

(c) $\mathrm{E}(|X_n - 0|^r) \;=\; n^{r\alpha - 1} \;\to 0$ iff $-\infty < \alpha < 1/r$.

2. (a) $\mathrm{E}(f_n(x)) \;=\; (2b_n)^{-1}(F(x+b_n) - F(x-b_n)) \to\; 0$ if $b_n \;\to\; 0$.

(b)

$$\begin{aligned}
\mathrm{var}(f_n(x)) &= (4b_n^2 n)^{-1}\mathrm{var}(I(x - b_n < X \leq x + b_n)) \\
&= (2nb_n)^{-1}\left(\frac{F(x+b_n) - F(x-b_n)}{2b_n}\right)(1 - (F(x+b_n) - F(x-b_n))) \\
&\to 0 \cdot f(x) \cdot 1 \;=\; 0 \;\text{ if } b_n \to 0,\, nb_n \to \infty.
\end{aligned}$$

(c) $f_n(x) - \mathrm{E}(f_n(x)) \;=\; (2nb_n)^{-1}\sum_{i=1}^n I(x - b_n < X_i \leq x + b_n) - \mathrm{E}(f_n(x))$,
so have mean 0, $\mathrm{var}(I(x - b_n < X_i \leq x + b_n)/\sqrt{2b_n}) \;\to\; f(x)$ as above, and by CLT and Slutsky $\sqrt{2nb_n}(f_n(x) - \mathrm{E}(f_n(x)) \;\to_d\; N(0, f(x))$ if $b_n \to 0$, $nb_n \to \infty$ as in (b).

(d) Need $\sqrt{2nb_n}(\mathrm{E}(f_n(x) - f(x))) \;=\; \sqrt{2nb_n}((F(x+b_n) - F(x-b_n))/(2b_n) \;-\; f(x)) \;\to\; 0$ so $b_n\sqrt{2nb_n} \to 0$ or $nb_n^3 \to 0$ should do it.

(e) Suppose we have the conditions for (d) and apply g' theorem to (c), with $g(y) \;=\; \sqrt{(y)}$, so $(g'(f(x)))^2 = 1/(4f(x))$ and

$$\sqrt{2nb_n}(\sqrt{f_n(x)} - \sqrt{f(x)}) \;\;\to_d\;\; N(0, (g'(f(x)))^2 f(x)) \;\;\equiv\;\; N(0, 1/4)$$

3. (a) With the change of variable suggested

$$I \;\;\equiv\;\; \int_1^\infty \frac{1}{x}\sin(2\pi x)dx \;\;=\;\; \int_0^1 \frac{1}{y}\sin(2\pi/y)dy$$

However, if $Y_1, \ldots, Y_n$ are i.i.d $U(0,1)$,

$$\begin{aligned}
\mathrm{E}\left(|\frac{1}{Y_1}\sin(2\pi/Y_1)|\right) &= \int_0^1 \frac{1}{y}|\sin(2\pi/y)dy \;\;=\;\; \int_1^\infty \frac{1}{x}|\sin(2\pi x)|dx \\
&= \sum_{k=3}^\infty \int_{(k-1)/2}^{k/2} \frac{1}{x}|\sin(2\pi x)|dx \;\;\geq\;\; \sum_{k=3}^\infty \frac{2}{k}\int_{(k-1)/2}^{k/2} |\sin(2\pi x)|dx \\
&= \sum_{k=3}^\infty \frac{2}{k}\frac{1}{\pi} \;\;=\;\; \infty
\end{aligned}$$

So the SLLN fails, and this will probably not be a good way to estimate I.

(b) Now

$$I_\alpha \;\;\equiv\;\; \int_1^\infty \frac{1}{x^\alpha}\sin(2\pi x)dx \;\;=\;\; \int_0^1 y^{\alpha-2}\sin(2\pi/y)dy$$

and in this case, if $Y \sim U(0,1)$

$$\mathrm{E}\left(|Y^{\alpha-2}\sin(2\pi/Y)|\right) \quad \leq \quad \mathrm{E}(Y^{\alpha-2}) \quad = \quad (\alpha-1)^{-1}\left[y^{\alpha-1}\right]_0^1 \quad < \quad \infty$$

is $\alpha > 1$. So by SLLN the estimator will converge $a.s.$ to I_α if $\alpha > 1$.

(c) Now

$$\mathrm{E}\left(|Y^{\alpha-2}\sin(2\pi/Y)|^2\right) \quad \leq \quad \mathrm{E}(Y^{2\alpha-4}) \quad = \quad (2\alpha-3)^{-1}\left[y^{2\alpha-3}\right]_0^1 \quad < \quad \infty$$

if $\alpha > 3/2$. So then, by the CLT, if $\alpha > 3/2$,

$$n^{\frac{1}{2}}(\widehat{I_{n,\alpha}} - I_\alpha) \quad \to_d \quad N(0, \sigma^2(\alpha))$$

where $\sigma^2(\alpha) \;=\; \mathrm{E}((Y^{2\alpha-4}(\sin(2\pi/Y))^2)\;\; I_\alpha^2$.

4. (a) $R = \overline{XY}/(\overline{X^2 Y^2})^{1/2}$.
By SLLN $\overline{X^2} \to_{a.s.} \mathrm{E}(X_i^2) = 1$, $\overline{Y^2} \to_{a.s.} \mathrm{E}(Y_i^2) = 1$, and $\overline{XY} \to_{a.s.} \mathrm{E}(X_i Y_i) = \rho$.
By CLT, $\sqrt{n}(\overline{XY} - \rho,\ \overline{X^2} - 1,\ \overline{Y^2} - 1)' \;\to_d\; N_3(0, V)$ where $V_{11} = \mathrm{var}(X_i Y_i) \;=\; \mathrm{E}(X^2Y^2) - \rho^2$, $V_{12} = \mathrm{Cov}(X_i^2,\ X_iY_i) \;=\; \mathrm{E}(X^3Y) - \rho$, etc.

(b) Consider $g(z_1, z_2, z_3) \;=\; z_1 z_2^{-1/2} z_3^{-1/2}$, so $R = g(\overline{XY},\ \overline{X^2},\ \overline{Y^2})$. Then

$$\left(\frac{\partial g}{\partial z_i}\right)|_{(\rho,1,1)} \quad = \quad ((z_2z_3)^{-\frac{1}{2}}, -\frac{1}{2}z_1z_2^{-\frac{3}{2}}z_3^{-\frac{1}{2}}, -\frac{1}{2}z_1z_2^{-\frac{1}{2}}z_3^{-\frac{3}{2}}) \quad = \quad (1, -\rho/2, -\rho/2)$$

So, by g' theorem $\sqrt{n}(R-\rho) \;\to_d\; (Z_1 - \frac{1}{2}\rho Z_2 - \frac{1}{2}\rho Z_3)$ where (Z_1, Z_2, Z_3) has the $N_3(0,V)$ dsn of (a).

(c) $X,\ Y$ indep $\Rightarrow \quad \rho = 0$, and $\mathrm{var}(Z_1) = E(X^2Y^2) - \rho^2 \;= 1$, so $\sqrt{n}R \to_d Z_1 \;\sim\; N(0,1)$.

(d) If (X_1, Y_i) Normal

$$\begin{aligned}
\mathrm{var}(XY) \quad &= \quad \mathrm{E}(X^2Y^2) - \rho^2 \quad = \quad \mathrm{E}(X^2\mathrm{E}(Y^2|X)) - \rho^2 \\
&= \quad \mathrm{E}(\rho^2X^4 + (1-\rho^2)X^2 - \rho^2) \quad = \quad 3\rho^2 + (1-\rho^2) - \rho^2 \;=\; 1 + \rho^2 \\
\mathrm{E}(X^3Y) - \rho \quad &= \quad \mathrm{E}(\rho X^4) - \rho \quad = \quad 2\rho \quad = \quad \mathrm{E}(XY^3) - \rho \\
\mathrm{E}(X^2Y^2) - 1 \quad &= \quad 2\rho^2, \quad \mathrm{var}(X^2) \;=\; \mathrm{var}(Y^2) \;=\; 2
\end{aligned}$$

then direct calculation of $(1, \rho/2, \rho/2)V(1, \rho/2, \rho/2)'$ gives $\sqrt{n}(R-\rho) \;\to_d\; N(0, (1-\rho^2)^2)$

(e) $g(x) = (1/2)(\log(1+x) - \log(1-x))$, $g'(x) = (1/2)((1+x)^{-1} + (1-x)^{-1}) = (2(1-x^2))^{-1}$.
So $\sqrt{n}(g(R) - \rho) \;\to_d\; N(0, 1/4)$ if (X_i, Y_i) is Normal.

5. (a) $\mathrm{E}(X_i - \mu)^k \;=\; \mu_k$ so, by SLLN, $B_k \to_{a.s.} \mu_k$.

(b) $\mathrm{var}((X_i - \mu)^k)\ \exists$ finite, since $\mu_{2k} < \infty$. Also $\mu_1 \;= \mathrm{E}(X_i - \mu) \;= 0$.
So by CLT $\sqrt{n}((B-1..., B_k)' - (0, \mu_2, ..., \mu_k)') \;\to_d\; N(0, V)$ where $V_{ij} \;=\; \mathrm{Cov}((X_l - \mu)^i, (X_l - \mu)^j) \;=\; \mu_{i+j} - \mu_i\mu_j$

(c) $nM_k \;=\; \sum_i (X_i - \overline{X_n})^k$, and expanding $(X_i - \overline{X_n})^k \;=\; ((X_i - \mu) - (\overline{X_n} - \mu))^k$ gives $M_k \;=\; \sum_0^k (-1)^{k-j} C(k:j) B_j B_1^{k-j}$, noting $B_1 = (\overline{X_n} - \mu)$. (Here $C(k:j)$ denotes the combinatorial coeff "k-choose-j".)
Now $B_1 \to_{a.s.} 0$, so $M_k \;\to_{a.s.}\; (-1)^{k-k}C(k:k)\lim(B_k).1 \;=\; 1$

(d)

$$
\begin{aligned}
M_i &= \sum_0^i (-1)^{i-j} C(i:j) B_j B_1^{i-j} &=& \; B_i \; - iB_{i-1}B_1 \; + \frac{1}{2} i(i-1) B_{i-2} B_1^2 \; + ... \\
(M_i - \mu_i) \; - (B_i - \mu_i - i\mu_{i-1}B_1) &&=& \; i\mu_{i-1}B_1 - iB_{i-1}B_1 \; + O_p(B_1^2) \\
&&=& \; -iB_1(B_{i-1} - \mu i - 1) \; + O_p(B_1^2) \\
\sqrt{n}((M_i - \mu_i) \; - (B_i - \mu_i - i\mu_{i-1}B_1)) &&=& \; (-i(B_1\sqrt{n}))(B_{i-1} - \mu_{i-1}) + (B_1\sqrt{n})O_p(B_1) \; \to_p \; 0
\end{aligned}
$$

since $B_1 \;\to_p\; 0$, $\sqrt{n}B_1 \;\to_d\; N(0,1)$, $(B_{i-1} - \mu_{i-1}) \;\to_p\; 0$.

(e) The limiting dsn of $\sqrt{n}(M_i - \mu_i)$ is the same as that of $\sqrt{n}(B_i - \mu_i - i\mu_{i-1}B_1)$ or $N_{k-1}(0, V^*)$ where now $i = (2, .., k)$ and

$$
\begin{aligned}
V_{ij}^* &= \text{Cov}((X-\mu)^{i+1} - (i+1)\mu_i(X-\mu),\; (X-\mu)^{j+1} - (j+1)\mu_j(X-\mu)) \\
&= \mu_{i+j+2} - (j+1)\mu_{i+2}\mu_j - (i+1)\mu_i\mu_{j+2} + (i+1)(j+1)\mu_i\mu_j\mu_2 - \mu_{i+1}\mu_{j+1}
\end{aligned}
$$

(f) $G_2 \to_{a.s.} \mu_4/\mu_2^2 \; - 3 \; = \; 0$ and $\mu_{2i+1} \; = \; 0$. Then $\sqrt{n}(M_2 - \mu_2,\; M_4 - \mu_4)$ cges to bivariate Normal, mean 0, variance $(\mu_4 - \mu_2^2, \mu_6 - \mu_2\mu_4, \mu_8 - \mu_4^2)$. Now let $g(x,y) = y/x^2$, so $g' = (-2\mu_4/\mu_2^3, 1/\mu_2^2)$, so limiting variance of $\sqrt{n}G_2$ is $(4\mu_4^2(\mu_4 - \mu_2^2) - 4\mu_4(\mu_6 - \mu_2\mu_4)\mu^2 + (\mu_8 - \mu_4^2)\mu_2^2)/\mu_2^6$. (Arithmetic not guaranteed.)