Stat 581 Homework 3: Outline Solutions, 2004

1. (a) Since the X_i are exchangeable, let $\text{var}(X_i) = \sigma^2$ and $\rho(X_i, X_j) = \rho$, $\forall\ i, j$. Then $\text{var}(X_1 + \ldots + X_n) = n\sigma^2(1 + (n-1)\rho)$. If $\rho < 0$, choose $n > 1 - 1/\rho$. Then the variance is < 0: contradiction. So $\rho \geq 0$.

(b) Conversely, suppose $0 \leq \rho \leq 1$. Let $X_i = Y + Z_i$ where Z_i are i.i.d. and indep of Y. Then the X_i are exchangeable, and $\rho = \text{var}(Y)/(\text{var}(Y) + \text{var}(Z))$, so $\text{var}(Y)$ and $\text{var}(Z)$ can be chosen to give any non-negative ρ. (For $\rho = 0$, $Y \equiv 0$. For $\rho = 1$, $Z \equiv 0$.)

2. $f(\mathbf{x}; \theta) = \prod_1^n (f(x_i; \theta) I(a(\theta) \leq x_i) I(b(\theta) \geq x_i)) = (\prod_1^n f(x_i; \theta)) I(a(\theta) \leq x_{(1)}) I(x_{(n)} \leq b(\theta))$

(a) $b(\theta) \equiv b$. If $f(x_i; \theta) = g(x_i)/h(\theta)$, $X_{(1)}$ is sufficient, by factorization thm.
Conversely, if $X_{(1)}$ is sufficient, then $f(\mathbf{x}; \theta) = g^*(\mathbf{x}) h^*(\theta, X_{(1)})\ \prod_1^n f(x_i; \theta)$. So h^* cannot depend on $X_{(1)}$, and $f(x_i; \theta) = g^*(x_i) h^*(\theta)$.

(b) If $f(x_i; \theta) = g(x)/h(\theta)$,

$$LR = \frac{f(\mathbf{x}; \theta)}{f(\mathbf{x}; \theta^*)} = \left(\frac{h(\theta^*)}{h(\theta)}\right)^n \frac{I(a(\theta) \leq X_{(1)})\ I(X_{(n)} \leq b(\theta))}{I(a(\theta^*) \leq X_{(1)})\ I(X_{(n)} \leq b(\theta^*))}$$

If a and b are both increasing, numerator becomes $I(b^{-1}(X_{(n)}) \leq \theta \leq a^{-1}(X_{(1)}))$.
Simiarly if a and b are both decreasing, it is $I(a^{-1}(X_{(1)}) \leq \theta \leq b^{-1}(X_{(n)})$.
In both cases this gives $(X_{(1)}, X_{(n)})$ min sufft. by the LR criterion.

(c) If a is increasing, but b decreasing, the indicator becomes $I(\theta \leq a^{-1}(X_{(1)}), \theta \leq b^{-1}(X_{(n)}))$ which is $I(\theta \leq \min(a^{-1}(X_{(1)}), b^{-1}(X_{(n)})))$ and $\min(a^{-1}(X_{(1)}), b^{-1}(X_{(n)}))$ is min sufft.
Similarly if a is decreasing, and b increasing, the indicator becomes $I(\theta \geq a^{-1}(X_{(1)}), \theta \geq b^{-1}(X_{(n)}))$ so the min sufft statistics is $\max(a^{-1}(X_{(1)}), b^{-1}(X_{(n)}))$.

(d) By (c), $\max(-X_{(1)}, X(n)) = \max(|X_{(1)}|, |X_{(n)}|)$ is sufficient.

(e) $X \sim U(\theta, \theta+1)$ where θ is an integer. If observe X then $\Pr(\lfloor X \rfloor = \theta) = 1$. So in a sample $\lfloor X_i \rfloor$ are a.s. equal and a.s. $= \theta$, and $\lfloor X \rfloor$ for any obs X is a strongly consistent estimator.

3. (a) Use the LR criterion, considering the LR for each π^l, $l = 1, ..., k$ against $\pi^{(0)}$ (wlog). Then the criterion reduces to the partition defined by

$$\sum_j (\pi_j^{(l)} - \pi_j^{(0)}) T_j, \text{ for } l = 1, ..., k$$

and hence to $(T_1, ..., T_k)$ since these $\pi^{(l)}$ are affinely independent.

(b) Here the natural parameters are $(\theta^{-1}, -\theta^{-2})$ corresponding to natural sufficient statistics $(\overline{X_n}, S^2)$, so have a curved family with $\pi_2 = -\pi_1^2$. So $k = 2$, and we can pick any 3 points on this parabola, spanning $\Re^2$. So $(\overline{X_n}, S^2)$ are min sufft.
However, there is no open rectangle, so we think $(\overline{X_n}, S^2)$ is probably not complete, and in fact it is not: $\text{E}(\overline{X_n}^2) = \theta^2(1+1/n)$, and $\text{E}(S^2) = (n-1)\theta^2$ (or something like this), so take $(n/(n+1))\overline{X_n}^2 - S^2/(n-1)$ to get something with expectation 0 $\forall\ \theta$.

4. (a) $\text{E}(Y_n - \mu)^2 = \int (y - \mu)^2 dF_n(y) \geq \epsilon^2 \Pr(|Y_n - \mu| > \epsilon)$.
For counterexample to converse, $P(Y_n = \sqrt{n}) = 1/n$, $P(Y_n = 0) = 1 - (1/n)$ will do. $Y_n \to_p 0$, but $\text{E}(Y_n^2) = 1$ for all n.

(b) $\text{var}(X_n - Y_n) = 2(1-\rho_n) \to 0$, so from (a) $X_n - Y_n \to_p 0$, so $X_n - Y_n \to_d 0$, but $X_n - Y_n \to_d X - Y$, so $X =_d Y$.

5. t_0 is s.t. $F_n^{-1}(t_0) \not\to F^{-1}(t_0)$, $|F_n(t_0) - F(t_0)| > \epsilon$ i.o.
(a) If $F_n^{-1}(t_0) < F^{-1}(t_0) - \epsilon$, $t_0 \leq F_n(F_n^{-1}(t_0)) \leq F_n(F^{-1}(t_0) - \epsilon \to F(F^{-1}(t_0) - \epsilon)$, which is contradiction (F^{-1} is left cts).
So $F_n^{-1}(t_0) > F^-1(t_0) + \epsilon$ i.o.

(b) Let $x_0 = F^{-1}(t_0)$, so $F_n^{-1}(t_0) > x_o + \epsilon$ i.o.
So $F(x_0 + \epsilon) \leq F(\liminf F_n^{-1}(t_0)) \leq F(x_0) \leq F(x_0 + \epsilon)$. So $F(x_0) = F(x_0 + \epsilon)$.

(c) The flat parts of F are the discontinuities of F^{-1}. $F_n^{-1}(t_0) \not\to F^{-1}(t_0) \Rightarrow F$ flat at $x_0 = F^{-1}(t_0)$. So there can be at most countably many such – assign a rational in each "gap".