Homework 2: Outline solutions: October 2004

1. Note that the event A_n i.o. is the event $\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$ and the complement of $\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k^c$.

Now if $\sum_1^{\infty} \Pr(A_k)$ is finite

$$\Pr(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k) \leq \Pr(\cup_{k=n}^{\infty} A_k) \leq \sum_{k=n}^{\infty} \Pr(A_k) \to 0$$

as $n \to \infty$.

Conversely, if $\sum_1^{\infty} \Pr(A_k) = \infty$, and A_k are independent, consider

$$\begin{aligned}\Pr(\cap_{k=n}^{\infty} A_k^c) &= \prod_{k=n}^{\infty} \Pr(A_k^c) = \prod_{k=n}^{\infty} (1 - \Pr(A_k)) \\ &\leq \prod_{k=n}^{\infty} \exp(\Pr(A_k)) = \exp(\sum_{k=n}^{\infty} \Pr(A_k)) = 0\end{aligned}$$

Hence, since $B_n = \cap_{k=n}^{\infty} A_k^c$ is an increasing sequence, $\Pr(\cup_n B_n) = \lim_n \Pr(B_n) = 0$, and the complement of this event (which is A_n i.o) has probability 1.

2. there are only three possibilities, so consider three points x_i for $i = 1, 2, 3$:
x_0 at which F is cts and strictly increasing
x_i at which F has a "jump": $F(x_1 - \epsilon) \to t_1 < F(x_1)$ as $0 < \epsilon \to 0$
x_2 at which F is "flat": $F(x_2 - \epsilon) = t_2 = F(x_2)$ for some $\epsilon > 0$.

(a) At points type x_0, x_1, F^{-1} is continuous. Suppose F^{-1} is not left-continuous at x_2.
So $\forall \delta > 0$, $\exists \epsilon > 0$ s.t. $F^{-1}(t_2 - \delta) < F^{-1}(t_2) - \epsilon$ so $F(x) \leq F(F^{-1}(t_2) - \epsilon)$, $\forall\, x < F^{-1}(t_2)$,
so $F(F^{-1}(t_2)) \leq t_2 - \delta$. Contradiction.

(b) $F^{-1}(F(x_0)) = x_0$, $F^{-1}(F(x_1)) = x_1$, $F^{-1}(F(x_2)) < x_2$.

(c) $F(F^{-1}(t_0)) = t_0$, $F(F^{-1}(t_1)) > t_1$, $F(F^{-1}(t_2)) = F(x_2 - \epsilon) = F(x_2) = t_2$.

(d) Suppose $F(x) \geq t$, then $F^{-1}(t) = \inf\{y : F(y) \geq t\} < x$.
Conversely, suppose $F^{-1}(t) \leq x$ or $x \geq \inf\{y : F(y) \geq t\}$ so $F(x) \geq F(y) \geq t$.

3. (a) If $k = 1$, $f_X(x; c) = cx^{-(c+1)} I(x > 1) = cx^{-1} I(x > 1) \exp(-c \log x)$, which is of required form with $\pi_1 = -c$ and $T_1(x) = \log\ x$.

(b) For varying k, the support depends on k, so it cannot be of exponential family form.

(c) $\Pr(Y \equiv \log\ X \leq y) = \Pr(X \leq e^y) = (1 - \exp(-c(y - \log k)))$ which is requd exponential cdf on $y > \log k$.

(d) This family of dsns for Y is a location-scale family with parameters $(\log k, c^{-1})$. Thus the group of transformations is $g_{a,b}(y) = a + by$, $b > 0$, and corresponding transformations $(\log\ k, c^{-1}) \to (a + b \log k, bc^{-1})$ or $g_{a,b}^*(k, c) = (e^a k^b, c/b)$. Thus the group of transformations on X is $g_{A,b}(x) = AX^b$, with corresponding parameter transformation $g_{A,b}^*(k, c) = (Ak^b, c/b)$, $A > 0$, $b > 0$.

4. (a) Mgf of X is $M_X(t) = \mathrm{E}(\exp(tX)) = \exp((k_0(\eta) - k_0(\sigma^2 t + \eta))/\sigma^2)$, so the cumulant generating function is $\sigma^{-2}(k_0(\eta) - k_0(\eta + \sigma^2 t))$.

(b) Differentiating r times, and setting $t = 0$, $\kappa_r(X) = \sigma^{2r-2} \kappa_r(Y)$

(c) $W = \sigma^2 Y$, so $f_W(w) = \sigma^{-2} f_Y(w/\sigma^2)$, which is not of the required form – note the term $\exp(k_0(\eta))$ remains unchanged, and is not multiplied by σ^{-2}. Scaling multiplies the rth moment by σ^{2r}. Not the same thing as multiplying the cumulants.

5. (a) Let $Z_i \equiv g_{a,b}(Y_i) = a + bY_i$, $(b > 0)$, then $Z_1, \ldots, Z_n$ are i.i.d uniform on $g^*_{a,b}(\theta_1, \theta_2) \equiv (a + b\theta_1,\ a + b\theta_2)$. (Location-scale transformation group.)

(b) As in any n-sample from a location-scale family, ratios of differences provide the invariants. A maximal invariant is $h = (h_1, \ldots, h_n)$, where $h_j = (Y_j - \overline{Y_n})/S_n(Y_1, \ldots, Y_n)$ where $\overline{Y_n}$ is sample mean, and $S_n(Y_1, \ldots, Y_n)$ is any constant multiple of the sample standard deviation – see Severini P.8.
A neater form for this problem is perhaps $(Y_i - Y_{(1)})/(Y_{(n)} - Y_{(1)})$, for $i = 2, ..., (n-1)$.
Note we have 2 parameters, so maximal invariant will have $n - 2$ components.