Homework 1: Outline solutions

1. (a) Draw a picture: easiest to separate into $x < y,\ v < 0$ and $x \geq y,\ v \geq 0$.
The the range of values (v, w) of (V, W) is

$$\{(v,w):\ -1 \leq v < 0,\ -v \leq w \leq 1\}\ \cup\ \{(v,w);\ 0 \leq v \leq 1,\ v \leq w \leq 1\}$$

(b) If transforming the density, again separate into $x > y$ and $x < y$. The e.g. on $x > y$ inverse transformation is $X = W, Y = W - V$, note this mapping is continuously differentable, and don't forget the Jacobian (even though it turns out to be 1).

Easier, is direct use of the distribution function

$$\begin{aligned} P(V \leq v,\ W \leq w) &= \Pr(X - Y \leq v,\ \max(X,Y) \leq w) \\ &= \Pr(Y \geq X - v, X \leq w, Y \leq W) \\ &= \begin{cases} w^2 - \frac{1}{2}(w-v)^2, & \text{if } 0 \leq v \leq w \leq 1 \\ \frac{1}{2}(v+w)^2, & \text{if } -1 \leq v < 0,\ 0 < -v \leq w \leq 1. \end{cases} \end{aligned}$$

Use your picture!
Differentiating w.r.t v and w gives the uniform density 1 on the range of (V, W).

(c) No, they cannot be independent since the ranges are not separable: we know $|v| \leq w$.

2. (a) The sequence of random variables V_n converges in distribution to $U(0,1)$ since

$$\Pr(V_n\ \leq\ x)\ =\ \lfloor nx \rfloor / n\ \ \rightarrow\ x \quad \text{as } n \rightarrow \infty$$

for $0 < x < 1$. (Draw a picture!)
For convergence in probability there is insufficient information to say. We do not know the joint dsn of the V_n nor even that they are defined on the same probability space.

(b) There are various ways to do this. This one comes from Ferguson.
Let $\epsilon > 0$ and $k > 2/\epsilon$ (an integer).
Since F is continuous, can choose x_j s.t. $F(x_j) = j/k,\ j = 1, ..., k-1$.
Since $F_n(x_j) \rightarrow F(x_j)$, $\exists\ N_j$ s.t. $|F_n(x_j) - F(x_j)|\ <\ 1/k$ for all $n > N_j$.
Now consider $n\ >\ N = \max(N_1, ..., N_{k-1})$, and $x_j \leq x < x_{j+1}$, with $x_k = \infty$:

$$\begin{aligned} F_n(x)\ &\leq\ F_n(x_{j+1})\ \leq\ F(x_{j+1}) + 1/k\ \leq\ F(x) + 2/k \\ F_n(x)\ &\geq\ F_n(x_j)\ \geq F(x_j) - 1/k\ \geq F(x) - 2/k \end{aligned}$$

Hence $|F_n(x) - F(x)|\ \leq\ 2/k\ <\ \epsilon$ for all $n > N$ and for all x.

3. (a) Checking the integral

$$\int_0^1 \int_0^1 f_\theta(v,w)\,dv\,dw\ =\ 1 \quad \text{for all real } \theta$$

we see that all we need is $f_\theta(v, w) \geq 0$ for all (v, w) in unit square, which by monotonicity of $(1 - 2v)$ will hold iff it holds at the corners (0,0), (0,1), (1,0) and (1,1) of the square, or for $|\theta| \leq 1$.

(b) The corresponding distribution function F_θ is given by

$$\begin{aligned} F_\theta(v,w)\ &=\ \int_0^v \int_0^w f_\theta(x,y)\,dx\,dy \\ &=\ vw(1\ +\ \theta(1-v)(1-w)) \end{aligned}$$

Note that

$$F_\theta(v,1) \;=\; v \quad \text{and} \quad F_\theta(1,w) \;=\; w,$$

so each of V and W has a uniform $U(0,1)$ marginal distribution.

(c) Now $\mathrm{E}(V) = \mathrm{E}(W) = 1/2$ and $\mathrm{var}(V) = \mathrm{var}(W) = 1/12$, Then direct integration gives

$$\mathrm{Cov}(V,W) \;=\; \int_0^1 \int_0^1 vw f_\theta(v,w)dvdw \;-\; (1/2)^2 \;=\; \theta/36$$

and hence

$$\rho(V,W) \;=\; \frac{\mathrm{Cov}(V,W)}{\sqrt{\mathrm{var}(V)\mathrm{var}(W)}} \;=\; \theta/3$$

Thus $|\rho(V,W)| \;\leq\; 1/3$, which limits the range of correlation between two $U(0,1)$ random variables that can be provided by this simple family.

4. (a) $\mathrm{E}(Y_n) \;=\; \theta/2$ and $\mathrm{var}(Y_n) \;=\; \theta^2/12$, hence immendiately by the CLT, V_n does converge in distribution, and it converges to $N(0,\theta^2/12)$.

(b) $\mathrm{Pr}(Y_n) \;>\; y) \;=\; 1 - y/\theta$, so $\mathrm{Pr}(\min(Y_n) \;>\; y) \;=\; (1-y/\theta)^n$.Then

$$\begin{aligned} P(W_n > w) \;&=\; \mathrm{Pr}(\min(Y_n) \;>\; y/n) \;=\; (1 - y/n\theta)^n \\ &\to\; \exp(-y/\theta) \quad \text{as } n \to \infty \end{aligned}$$

Thus W_n does converge in distribution. It converges to an exponential random variable with mean θ.