Stat 581 Homework 8: Due December 1, 2004

1. Suppose that $X_1, ..., X_n$ are i.i.d from a distribution with

$$\mathrm{E}(X_i) = \theta, \quad \mathrm{var}(X_i) = 1, \quad m_4 = \mathrm{E}(X_i^4) < \infty, \text{ and } \mu_4 = \mathrm{E}(X_i - \theta)^4$$

Let $(T_{1,n}) = n^{-1} \sum_1^n X_i^2 \ - 1$, $T_{2,n} = (n^{-1} \sum_1^n X_i)^2 - n^{-1}$.
(a) Show that both sequences of estimators are unbiased for θ^2, and determine the variances of their standardized asymptotic distributions.
(b) Show that if the distribution is symmetric, $\mu_4 = m_4 - 6\theta^2 - \theta^4$. Hence show that, in this case, the A.R.E. of $(T_{2,n})$ relative to $(T_{1,n})$ is ≥ 1.
(c) Give an example of a distribution for which the A.R.E. is ≤ 1.

2. Let $X(u) \sim N(\alpha + \beta u, \sigma^2)$, where $0 \leq u \leq 1$, σ^2 is known, and α and β are unknown. A design for this problem consists of choosing numbers $u_1, ..., u_n$ at which to observe $X(u_1), ..., X(u_n)$. The analysis of the data is based on the MLE/least-quares estimator $\tilde{\beta} = \sum_i (u_i - \overline{u}) X_i(u) / \sum_i (u_i - \overline{u})^2$.
(a) Determine the information matrix for (α, β) based on the sample.
(b) Assume, for simplicity than n is even. Show that a design that puts $n/2$ points at $u = 0$ and $n/2$ at $u = 1$ maximized the determinant of the information matrix and minimizes the variance of $\tilde{\beta}$.
(c) A competing design has $u_i = (i-1)/(n-1)$. Find the relative efficiency of the estimator $\tilde{\beta}$ from this design, as compared to the design in (b).
(Reminder: $\sum_1^n i^2 = n(n+1)(2n+1)/6$.)
(d) Which design would you prefer to use, and why?

3. Suppose that $X_1, ..., X_n$ are i.i.d. Poisson with mean λ, $Y_1, ..., Y_n$ are i.i.d Poisson with mean μ, and that the two samples are independent.
(a) Derive the likelihood ratio test of the hypothessis $\lambda = 2\mu$.
(b) Derive the Rao (Score) test of the hypothesis $\lambda = 2\mu$.

4. Continuing 3, derive the Wald test of the hypothesis $\lambda = 2\mu$ in the parmetrization (β, μ) where
(a) $\beta = \lambda - 2\mu$; (b) $\beta = (\mu/\lambda)$; (c) $\beta = (\lambda/\mu)$; (d) $\beta = \log(\mu/\lambda)$.

5. Consider a 2×2 table, so that $(X_{11}, X_{12}, X_{21}, X_{22})$ is multinomial with index equal to the sample size n, and probabilities $(p_{11}, p_{12}, p_{21}, p_{22})$ with $p_{11} + p_{12} + p_{21} + p_{22} = 1$. This may be conveniently parametrized as $\theta = (p_{1\cdot}, p_{\cdot 1}, \psi)$ where $p_{1\cdot} = p_{11} + p_{12}$, $p_{\cdot 1} = p_{11} + p_{21}$ and $\psi = \log(p_{21} p_{12} / p_{11} p_{22})$. You may use the fact that $\psi = 0$ if and only if independence holds in the 2×2 table: that is $p_{ij} = p_{i\cdot} p_{\cdot j}$ for $i, j = 1, 2$.

(a) Find the MLE $\widehat{\psi_n}$ of ψ.

(b) Show that $\widehat{\psi_n}$ is asymptotically Normal, and the variance of the asymptotic Normal distribution is $\sum \sum_{i,j=1,2} (p_{ij})^{-1}$.

(c) Show that the usual test statistic for testing independence in a 2×2 table

$$Q_n \;=\; \sum_{i=1,2} \sum_{j=1,2} (X_{ij} - E_{ij})^2 / E_{ij}$$

where $E_{ij} = X_{i\cdot} X_{\cdot j} / n$, has an asymptotically χ_1^2 distribution under $\psi = 0$.

(d) Find the limiting value of $n^{-1} Q_n$ under a general value of (p_{ij}).

(e) **Assume** the usual theory holds under local alternatives, $\psi_n = s n^{-\frac{1}{2}}$, so that then Q_n is asymptotically $\chi_1^2(\delta)$, where $\delta = p_{,\cdot}(1 - p_{1\cdot}) p_{\cdot 1} (1 - p_{\cdot 1}) s^2$. (**Do not show this.**) Suppose that $n = 30$ and a test of size $\alpha = 0.02$ is used. Show how you would use the result to approximate the power of the test if the true (p_{ij}) were (0.3, 0.2, 0.1, 0.4).