Stat 581 Homework 6: Due November 17, 2004

1.(a) (TPE P.139, No. 5.23) Let $X_1, \ldots, X_n$ be i.i.d with density $f(x;\theta)$ which is strictly positive for all x, Show that the variance of any unbiased estimator T of θ satisfies

$$\text{var}_{\theta_0}(T) \quad \geq \quad \frac{(\theta - \theta_0)^2}{\left(\int_{-\infty}^{\infty} \frac{f(x;\theta)^2}{f(x;\theta_0)}\right)^n \; - \; 1} \qquad \theta \neq \theta_0.$$

(b) (TPE P.140, No. 5.24) If $X_1, \ldots, X_n$ are i.i.d $\sim N(\theta, \sigma^2)$, where σ is known, and θ is known to take one of the values $0, \;\pm 1, \;\pm 2, \pm 3 \ldots$, show that any unbiased estimator of the restricted parameter θ satisfies

$$\text{var}_{\theta_0}(T) \quad \geq \quad \frac{\Delta^2}{\exp(n\Delta^2/\sigma^2) \; - \; 1} \quad \equiv \quad h(\Delta) \quad \Delta \neq 0.$$

where $\Delta = (\theta - \theta^0)$, and hence $\text{var}_{\theta_0}(T) \;\geq\; \sup_{\Delta \neq 0} h(\Delta) \;=\; 1/(\exp(n/\sigma^2) \;-\; 1)$.

2. Continuing the same question as #1 (b)
(a) Show that the MLE, $\widehat{\theta_n}$ of θ is the integer closest to $\overline{X_n}$, and that this MLE is unbiased.
(b) Show that if $n \geq 15.5\sigma^2$, them $P(\widehat{\theta_n} = \theta) \geq 0.95$
(c) Show that $\widehat{\theta_n}$ is consistent.
(d) Show that $\text{var}(\widehat{\theta_n}) \;=\; 2\sum_1^\infty (2j-1)\;(1 - \Phi((j - \frac{1}{2})n^{\frac{1}{2}}/\sigma)$.
(Note: This can be used to show that the variance (like the bound) decays exponentially in n: TPE P.140)

3. Let $(X_i, Y_i), i = 1, ..., n$ be i.i.d. bivariate normal, with $\text{E}(X_i) = \text{E}(Y_i) = 0$, $\text{var}(X_i) = \text{var}(Y_i) = 1$ and $\text{Cov}(X_i, Y_i) = \rho$. Find the Cramer-Rao lower bound on the variance of unbiased estimators of ρ. Does the sample correlation r achieve this bound asymptotically? Why/why not?

4. (a) (Severini, Exx 3.10, P.103) Let $Y_1, \ldots, Y_n$ be i.i.d. with Gamma density

$$f_Y(y; \alpha, \beta) \quad = \quad \beta^\alpha y^{\alpha - 1} \exp(-\beta y) I_{(0,\infty)}(y)/\Gamma(\alpha)$$

where α and β are unknown parameters. Find the Fisher information matrix.

(b) (Severini, Exx, 3.11, P.104) For the model of part (a), find a parameter ϕ that is orthogonal to α and find a parameter η that is orthogonal to β.

5. (a) Suppose $X_1, ..., X_n$ are i.i.d $N(\theta, 1)$, so that $I_1(\theta) = 1$. Let $T_n = \overline{X_n}$ if $|\overline{X_n}| > n^{-1/4}$ and $T_n = a\overline{X_n}$ if $|\overline{X_n}| \leq n^{-1/4}$.
Show that $n^{\frac{1}{2}}(T_n - \theta) \rightarrow_d N(0, V^2(\theta))$ where $V^2(\theta) = 1$ if $\theta \neq 0$, and $V^2(\theta) = a^2$ if $\theta = 0$.
Choosing $|a| < 1$, then gives $V^2(\theta) < 1/I_1(\theta)$ at $\theta = 0$. Why does this not violate the CRLB theory?

(b) Suppose that $X_1, ..., X_n$ are i.i.d., with each X_i being $N(\theta, 1)$ with probability $\frac{1}{2}$ and $N(-\theta, 1)$ with probability $\frac{1}{2}$. Evaluate the information about θ. What happens when $\theta = 0$? What does this suggest to you about how you should estimate θ?