Stat 581 Homework 4: Due October 27, 2004

You may quote standard results and theorems, but should be specific about which one(s) you are citing, and why.

1. Let X_n $n = 1, 2, 3, ...$ be independent random variables defined on a common probability space Ω and such that

$$P(X_n = n^\alpha) \;=\; \frac{1}{n} \quad \text{and} \quad P(X_n = 0) \;=\; 1 - \frac{1}{n} \qquad n = 1, 2, ...$$

where α is a constant. Find the values of α, $-\infty < \alpha < \infty$, for which

(a) X_n converges to 0 in probability,

(b) X_n converges to 0 a.s.

(c) X_n converges to 0 in r th moment, for given $r > 0$.

(Hint: See Homework 2, #1)

2. Let $X_1, ..., X_n$ be i.i.d. with continuous density f. Let

$$f_n(x) \;=\; \frac{F_n(x + b_n) - F_n(x - b_n)}{2b_n}$$

where F_n is the empirical distribution function of $X_1, ..., X_n$. Thus f_n can be thought of as an empirical density estimate.

(a) Show that $\mathrm{E}(f_n(x)) \;\to\; f(x)$ if $b_n \to 0$.

(b) Show that $\mathrm{var}(f_n(x)) \;\to 0$ if $b_n \to 0$ and $nb_n \to \infty$.

(c) Show that (under suitable conditions) $(2nb_n)^{\frac{1}{2}}(f_n(x) - \mathrm{E}(f_n(x)) \;\to_d\; N(0, f(x))$

(d) Find conditions under which $\mathrm{E}(f_n(x))$ can be replaced by $f(x)$ in (c).

(e) Find the limiting distribution of $(f_n(x))^{\frac{1}{2}}$ suitably normalized.
Note: $(f_n(x)^{\frac{1}{2}} - f(x)^{\frac{1}{2}})$ is called a *rootogram*.

3. Based on Ferguson. P.24 # 4, and JAW Problems 3 #1
(a) Ferg. P.24 #4: Show that the SLLN fails, and hence that $\widehat{I_n}$ is likely not a good estimator of I.

(b) Generalize the integral I of (a) to

$$I_\alpha \;=\; \int_1^\infty x^{-\alpha} \sin(2\pi x) dx$$

Construct the corresponding estimator $\widehat{I_{n,\alpha}}$, using the same change-of-variable as in (a). For what values of α will the estimator $\widehat{I_{n,\alpha}}$ converge to I_α a.s.
Hint: bound $g(y)\sin(2\pi/y)$ by $g(y)$.

(c) For what values of α will $n^{\frac{1}{2}}(\widehat{I_{n,\alpha}} - I_\alpha)$ converge in distribution, and to what? (Same hint: you need not find the limiting variance explicitly.)

4. Let $(X_i, Y_i)'$, $i = 1, ..., n$ be i.i.d. bivariate r.v.s with mean $(0,0)'$ and $\text{var}(X_i) = \text{var}(Y_i) = 1$, $\text{Cov}(X_i, Y_i) = \rho$.
Let S_{ZW} denote $n^{-1}\sum_{i=1}^n Z_i W_i$, and $R = S_{XY}/\sqrt{S_{XX}S_{YY}}$. We investigate R as an estimator of ρ. Assume $\text{E}(X^4) < \infty$, $\text{E}(Y^4) < \infty$.

(a) Show that

$$\sqrt{n}(S_{XY} - \rho, S_{XX} - 1, S_{YY} - 1)' \quad \rightarrow_d \quad (Z_1, Z_2, Z_3)' \sim N_3(0, \Sigma)$$

where $\Sigma_{11} = \text{E}(X^2Y^2) - \rho^2$, $\Sigma_{12} = \text{E}(X^3Y) - \rho$, $\Sigma_{22} = \text{E}(X^4) - 1$, $\Sigma_{23} = \text{E}(X^2Y^2) - 1$.

(b) Show that $n^{\frac{1}{2}}(R - \rho) \rightarrow_d (Z_1 - \rho(Z_2 + Z_3)/2)$.

(c) Show that if X_i and Y_i are independent then $n^{\frac{1}{2}}(R - \rho) \rightarrow_d N(0, 1)$ regardless of the underlying distribution (subject to the inital assumptions $\text{E}(X^4) < \infty$ etc.).

(d) Show that if (X_i, Y_i) is bivariate Normal, $\Sigma_{11} = 1 + \rho^2$, $\Sigma_{12} = 2\rho$, $\Sigma_{22} = 2$, $\Sigma_{23} = 2\rho^2$,
and hence that $n^{\frac{1}{2}}(R - \rho) \rightarrow_d N(0, (1 - \rho^2)^2)$.

(e) Suppose (X_i, Y_i) is bivariate Normal, and let $g(x) = \frac{1}{2}\log((1 + x)/(1 - x))$, $V = g(R)$, $\xi = g(\rho)$.
Show that, if (X_i, Y_i) is bivariate Normal, $n^{\frac{1}{2}}(V - \xi) \rightarrow_d N(0, 1)$.

5. Let $X_1, ..., X_n$ be i.i.d. with mean μ and k th central moment $\mu_k = \text{E}((X_1 - \mu)^k)$, and assume $\mu_{2k} < \infty$.
Define $B_k = n^{-1}\sum_{i=1}^n (X_i - \mu)^k$, and $M_k = n^{-1}\sum_{i=1}^n (X_i - \overline{X_n})^k$, where $\overline{X_n} = n^{-1}\sum_{i=1}^n X_i$.

(a) Show $B_k \rightarrow \mu_k$ a.s.

(b) Show that $n^{\frac{1}{2}}((B_1, ..., B_k)' - (0, \mu_2, ..., \mu_k)'$ is asymptotically (jointly) normal with mean $\mathbf{0}$ and variance $\mathbf{V}$ where $V_{ij} = \mu_{i+j} - \mu_i\mu_j$.

(c) Show that $M_k = \sum_{j=0}^k \frac{k!}{j!(k-j)!}(-1)^{k-j} B_j B_1^{k-j}$, and deduce that $M_k \rightarrow \mu_k$ a.s.

(d) Show that

$$n^{\frac{1}{2}}((M_i - \mu_i) \; - \; (B_i - \mu_i - i\mu_{i-1}B_1))$$

converges in probability to 0 as $n \rightarrow \infty$.

(e) Deduce that $n^{\frac{1}{2}}(M_2 - \mu_2, ..., M_k - \mu_k)'$ is asymptotically Normal $N_{k-1}(\mathbf{0}, \Sigma)$ where
$\sigma_{ij} = \mu_{i+j+2} - (i + 1)\mu_i\mu_{j+2} - (j + 1)\mu_{i+2}\mu_j + (i + 1)(j + 1)\mu_i\mu_j\mu_2 - \mu_{i+1}\mu_{j+1}$.

(f) Determine the asymptotic distribution of $G_2 \equiv (M_4/M_2^2 - 3)$, in the case where X_i has a symmetric distribution having $\mu_4/\mu_2^2 = 3$. (For example, if X_i are Gaussian.)