**Stat 581 Homework 3: Due October 20, 2004**

1. A sequence of real-valued random variables $X_1, X_2, ...,$ is exchangeable if, for any finite $k$, $(X_1, ..., X_k)$ has the same distribution as $(X_{\pi(1)}, ..., X_{\pi(k)})$ where $(\pi(j), j = 1, ...k)$ is any permutation of $(1, 2, .., k)$.

(i) Consider an infinite sequence $X_1, X_2...,$ of exchangeable random variables having finite second moments. Show that the correlation $\rho(X_i, X_j) \geq 0$ for all $i, j$.
(Hint: consider $\mathrm{var}(X_1 + X_2 + .... + X_k)$.)

(ii) Given any $\rho$, $0 \leq \rho \leq 1$, show that there exists an exchangeable sequence such that $\rho(X_i, X_j) = \rho$ for all $i, j$.
(Hint: consider sequences of the form $X_i = Y + Z_i$.)

2. Let $f(x; \theta)$ be a density on $a(\theta) \leq x \leq b(\theta)$, where (for (a)-(c)) $\theta$ takes values in a non-degenerate interval. and let $X_1, ..., X_n$ be a sample from this density.

(a) Suppose $b(\theta) \equiv b$. Show that $X_{(1)} = \min(X_i)$ is sufficient if and only if $f(x; \theta) = g(x)/h(\theta)$.

(b) Suppose $f(x; \theta) = g(x)/h(\theta)$. Show that if $a(\theta)$ and $b(\theta)$ are both increasing, or both decreasing, as functions of $\theta$, then $(X_{(1)}, X_{(n)}) = (\min(X_i), \max(X_i))$ is minimal sufficient.

(c) Suppose $f(x; \theta) = g(x)/h(\theta)$. If $a(\theta)$ and $b(\theta)$ are both monotone, but one is increasing and the other decreasing, find a one-dimensional sufficient statistic.

(d) If the density $f$ is that of a uniform $U(-\theta, \theta)$ r.v. $(\theta > 0)$, find a one-dimensional sufficient statistic.

(e) If the density $f$ is $U(\theta, \theta + 1)$, where $\theta$ is an integer, show that any observation is sufficient and exhibit a strongly consistent estimator of $\theta$.

3. In class we noted that the existence of an open rectangle of full dimension within the natural parameter space of an exponential family is a sufficient condition for completeness, and hence for minimal sufficiency of the natural sufficient statistics. However, this is too strong a condition to be useful. The following gives a more useful result that provides for the non-completeness of the natural minimal sufficient statistics in many curved exponential families.

(a) Consider an $n$-sample from an exponential family, with natural parameters $(\pi_1, ..., \pi_k)$ and natural sufficient statistics $(T_1, ..., T_k)$, $T_j = \sum_1^n t_j(X_i)$, with no affine relationships among the $T_j$. Show that, if the parameter space contains $k+1$ points $\pi^l$, $l = 0, .., k$ which span $\Re^k$ in the sense that they do not belong to an affine subspace of $\Re^k$, then $(T_1, ..., T_k)$ is minimal sufficient.

(b) Consider $X_1, ..., X_n$ i.i.d. from a Normal dsn $N(\theta, \theta^2)$. What is the minimal sufficient statistic? Is it complete?

4 (a) (Severini P.69, 2.2) Show that if the sequence of random variables $Y_1, Y_2, ...$ converges in quadratic mean to $\mu$, then the sequence converges in probability. Give an example to show that the converse is not true.

(b) (Severini P.69, 2.3) Let $(X_n)$ and $(Y_n)$ each be a sequence of real-valued random variables each with mean 0 and standard deviation 1. Suppose that, as $n \to \infty$, $X_n \to_d X$ and $Y_n \to_d Y$. Let $\rho_n$ be the correlation between $X_n$ and $Y_n$. Show that if $\rho_n \to 1$ as $n \to \infty$, then $X$ and $Y$ have the same distribution.

5. For distribution functions $F_n$ $(n = 1, 2, 3...)$ and $F$, let $F_n(x) \to F(x)$, $-\infty < x < \infty$. Suppose $t_0$ is s.t. $F_n^{-1}(t_0) \not\to F^{-1}(t_0)$. Choose $\epsilon$ s.t. $|F_n^{-1}(t_0) - F^{-1}(t_0)| > \epsilon$ for infinitely many $n$.

(a) Show that $F_n^{-1}(t_0) > F^{-1}(t_0) + \epsilon$ for infinitely many $n$
(Hint: show $F_n^{-1}(t_0) < F^{-1}(t_0) - \epsilon$ i.o. is impossible.)

(b) Deduce that $t_0 = F(F^{-1}(t_0)$ and that $F$ is therefore flat in a right-neighborhood of $F^{-1}(t_0)$.
(c) Show that there are at most countably many points in the set

$$\{t:\ 0 < t < 1,\ F_n^{-1}(t) \not\to F^{-1}(t), n \to \infty\}$$

Hint: a non-decreasing function has at most countably many discontinuities; why?