Stat 581 Homework 2: Due October 13, 2004

1. Show that for independent events A_n, $n = 1, 2, \ldots$, A_n occurs infinitely often if and only if $\sum_n P(A_n) = \infty$. (see Ferg, P12, #4)

2. Let F be a cdf, and for $0 < t < 1$ define $F^{-1}(t) = \inf\{x : F(x) \geq t\}$. Show that

(a) F^{-1} is left-continuous

(b) $F^{-1}(F(x)) \leq x$, $-\infty < x < \infty$

(c) $F(F^{-1}(t)) \geq t$, $0 < t < 1$

(d) $F(x) \geq t$ if and only if $x \geq F^{-1}(t)$.

3. The cdf of a r.v. X with a *Pareto distribution* with parameters (c, k) $(c > 0, k > 0)$ is given by

$$F_X(x) \quad = \quad P(X \leq x) \quad = \quad (1 - (k/x)^c) \qquad \text{on } x > k.$$

(a) Show that if $k = 1$, the family for varying c is a one-parameter exponential family, and identify the natural parameter and natural statistic.

(b) For varying (c, k), $(c > 0, k > 0)$ is the family an exponential family? Why/why not?

(c) Show that $\log X$ has an exponential distribution on $(\log k, \infty)$.

(d) Hence or otherwise, show that the Pareto family for varying $c > 0$ and $k > 0$ is a group family.

4. (Severini P. 25 1.6) A family of distributions that is closely related to the exponential family is the family of exponential dispersion models. Suppose that a scalar random variable X has density on some set A of the form

$$p(x; \eta, \sigma^2) \quad = \quad \exp((\eta x + k_0(\eta))/\sigma^2 \; + \; S(x, \sigma^2)) I_A(x), \qquad \eta \in H$$

where for known σ^2 the density p satisfies the conditions of a one-parameter exponential family distribution and H is an open set. The set of density functions $\{p(\cdot; \eta, \sigma^2 > 0\}$ is said to be an exponential dispersion model.
(a) Find the cumulant generating function of X (i.e. the log of the moment generating function).
(b) Suppose that a random variable Y has the density $p(\cdot; \eta, 1)$. That is, it has the same density as X, except σ^2 is known to be 1. Find the cumulants of X in terms of the cumulants of Y.
(c) Let $W = \sigma^2 Y$. Is the distribution of W an exponential dispersion model?

5. (Severini, P. 25 1.7) Let $Y_1, \ldots, Y_n$ denote independent identically distributed random variables each uniformly distributed on the interval (θ_1, θ_2), $\theta_1 < \theta_2$.
(a) Show that this is a transformation model and identify the group of transformations. Show the correspondence between the parameter space and the transformations,
(b) Find a maxmal invariant statistic. That is, a statistic $T(Y_1, \ldots, Y_n)$ whose value is unchanged by the transformations in the group, and such that any other invariant statistic is a function of T.