Final Exam: Stat 581: Fall 2004
SMI 405: 8.30-10.20, Monday Dec 13, 2004

There are four questions on this exam: You may attempt all four questions.
The questions are of equal weight.
This is a closed-book, closed-notes exam.

1. (a) Suppose $\epsilon_1, \epsilon_2, ...$ are independent and identically distributed random variables, each having same mean μ and variance σ^2. Define X_n as the autoregressive sequence

$$\begin{aligned} X_1 &= \epsilon_1 \\ X_n &= \beta X_{n-1} + \epsilon_n \qquad \text{for } n = 2, 3, ... \end{aligned}$$

where $-1 < \beta < 1$. Let $\overline{X_n} = n^{-1} \sum_{i=1}^{n} X_i$.

(a) Show that

$$\begin{aligned} X_n &= \sum_{j=1}^{n} \epsilon_j \beta^{n-j} \quad \text{and} \\ (1-\beta)\overline{X_n} &= n^{-1} \sum_{j=1}^{n} \epsilon_j - \beta X_n / n \end{aligned}$$

(b) Show that $\overline{X_n}$ converges to $\mu/(1-\beta)$ in quadratic mean.
(Hint: Show E($\overline{X_n}$) converges to $\mu/(1-\beta)$.)
(c) Show that

$$\sqrt{n}((1-\beta)\overline{X_n} - \mu) - \sqrt{n}(n^{-1} \sum_{j=1}^{n} \epsilon_j - \mu)$$

converges to 0 in probability.
(d) Show that

$$\sqrt{n}\left(\overline{X_n} - \frac{\mu}{1-\beta}\right) \to_d Z \sim N\left(0, \frac{\sigma^2}{(1-\beta)^2}\right)$$

2. Let X_i be i.i.d. with mean μ variance σ^2 and finite fourth central moment $\mathrm{E}(X_i - \mu)^4 = \mu_4 = (3+K)\sigma^4 < \infty$. Let $\overline{X_n} \equiv n^{-1} \sum_{i=1}^{n} X_i$.
(a) Let $Y_i \equiv (X_i - \mu)^2$, $\overline{Y_n} \equiv n^{-1} \sum_{i=1}^{n} Y_i$. Show that $n^{\frac{1}{2}}(\overline{Y_n} - \sigma^2) \to_d N(0, (2+K)\sigma^4)$.
(b) Let $S^2 \equiv (n-1)^{-1} \sum_{1=1}^{n} (X_i - \overline{X_n})^2$.
Show that $S^2 \to_{a.s.} \sigma^2$, and that $n^{\frac{1}{2}}(S^2 - \sigma^2) \to_d N(0, (2+K)\sigma^4)$.
(c) Show that $n^{\frac{1}{2}}(\overline{X_n} - \mu)/S \to_d N(0,1)$, so that validity of the t-test is robust against departures from Normality of the X_i.
(d) Suppose $\sigma = \sigma_0$ is tested against the alternative $\sigma > \sigma_0$ by rejecting the null hypothesis if $((n-1)S^2 > c_n \sigma_0^2$ where $c_n = G^{-1}(1-\alpha)$ where G is the cdf of a χ^2_{n-1}. Show that the probability of rejecting $\sigma = \sigma_0$, when this is true, is approximately $1 - \Phi((n/(2+K))^{1/2}(c_n/(n-1) - 1))$, where $\Phi()$ is the $N(0,1)$ cdf.
(e) By considering the case when X_i are $N(\mu, \sigma^2)$ show that $\sqrt{n/2}(c_n(n-1)^{-1} - 1) \to \Phi^{-1}(1-\alpha)$. Hence show that, when $\sigma = \sigma_0$, in general the probability of rejecting under the test (d) converges to $1 - \Phi((1+K/2)^{-\frac{1}{2}}\Phi^{-1}(1-\alpha))$. Deduce that this test is not robust against departures from Normality for which $K \neq 0$.

3. Suppose $X_1, X_2, ...$ are independent and identically distributed from a Normal distribution $N(\theta, \theta^4)$ with mean θ and variance θ^4.

(a) Find the minimal sufficient statistic for θ. Is it complete? Why?/Why not?

(b) Let $\widehat{\theta_n}$ be the maximum likelihood estimator of θ based on $(X_1, ..., X_n)$. Find its appropriately normalized non-degenerate limiting distribution as $n \to \infty$. **Do NOT attempt to find $\widehat{\theta_n}$.**

(c) Show that $\overline{X_n} = n^{-1} \sum_1^n X_i$ is a weakly consistent estimator of θ, and give its appropriately normalized non-degenerate limiting distribution as $n \to \infty$.
Is $\overline{X_n}$ an asymptotically efficient estimator? Why?/Why not?

(d) Hence or otherwise, find a consistent and asymptotically efficient estimator of θ with a closed-form expression.

4. Let $X_i,\ i = 1, ...m$ be i.i.d exponential with mean μ: $f_\mu(x) = \mu^{-1} \exp(-x/\mu) I_{(0,\infty)}(x)$. Let $Y_j,\ j = 1, ..., n$ be i.i.d exponential with mean λ, with each Y_j also independent of any X_i. Suppose the parameter of interest is $\theta = \mu/\lambda$.

(a) Compare the information about θ available from the two samples, with that available when λ is known.

(b) It is desired to test the hypothesis $\theta = 3$. Derive any **one** of the three asymptotic likelihood-based tests of this hypothesis. Determine your chosen test statistic and specify the test as explicitly as you can. State why you prefer the test that you have chosen.