Stat 581 Homework 7: Due November 26, 2003

1. Let $X_1, ..., X_n$ be i.i.d. $\sim U(0, \theta)$
(a) Find the MLE $\widehat{\theta_n}$ of θ.
(b) Show that $\widehat{\theta_n}$ is biased.
(c) Show that $\widehat{\theta_n}$ is consistent.
(d) Show that $\widehat{\theta_n}$ is unbiased in the limit as $n \to \infty$.
(e) Find constants k_n and non-degenerate dsn H s.t. $k_n(\widehat{\theta_n} - \theta) \to_d Z \sim H$ as $n \to \infty$.
(f) Show that $Z \sim H$ does not have mean 0. Hence $\widehat{\theta_n}$ is not asymptotically unbiased, in the sense of TPE.

2. Let $(X_i, Y_i), i = 1, ..., n$ be i.i.d. bivariate normal, with $\mathrm{E}(X_i) = \mathrm{E}(Y_i) = 0$, $\mathrm{var}(X_i) = \mathrm{var}(Y_i) = 1$ and $\mathrm{Cov}(X_i, Y_i) = \rho$. Find the likelihood equation for estimation of ρ. Show that it always (with probability 1) has at least one solution in (-1,1), and that the solution is unique for large enough values of ρ.

3. Suppose that $(U_{(1)}, ..., U_{(n)})$ are the order statistics of a a sample from a uniform $U(0,1)$ distribution. Suppose that $V_i,\ i = 1, ..., (n+1)$ are i.i.d standard exponentials on $(0, \infty)$– that is $f_V(v) = e^{-v} I_{(0,\infty)}(v)$. Define $W_k = \sum_{i=1}^k V_i$.
(a) Show that $(W_j/W_{n+1}; j = 1, ..., n)$ is independent of W_{n+1}
and $(W_j/W_{n+1}; j = 1, ..., n) =_d (U_{(j)}; j = 1, ..., n)$.
(Hint: Consider the conditional dsn of $(W_j/W_{n+1}; j = 1, ..., n)$ given W_{n+1}.)
(b) Use (a) to show that $U_{(\lfloor np \rfloor)}$ converges a.s. to p.
(c) Use (a) to show that $(U_{(r)}, U_{(s)})$, appropriately standardized, converges to a bivariate Normal dsn with the variance covariance matrix of Brownian bridge.
(d) Use (a) to show that if $r < s < t$, $U_{(r)}$ and $U_{(t)}$ are independent given $U_{(s)}$.

4. (a) Suppose $X_1, ..., X_n$ are i.i.d. $U(\theta - \psi, \theta + \psi)$. Find the minimal sufficient statistic for (θ, ψ) and the MLE of (θ, ψ).

(b) Suppose $X_1, ..., X_n$ are i.i.d. double exponential, each with pdf $f_\theta(x) = \frac{1}{2}\exp(-|x - \theta|)$. Find the minimal sufficient statistic and the MLE of θ

5. Suppose $X_1, ..., X_n$ are i.i.d. from a distribution symmetric about a location parameter θ and with a strictly positive density over the range of X_i. It is proposed to estimate θ by the average of the pth and $(1-p)$th sample quantiles, $T_n^{(p)} = \frac{1}{2}(F_n^{-1}(p) + F_n^{-1}(1-p))$, where F_n is the empirical distrbution function of $X^{(n)} = (X_1, ..., X_n)$.
(a) Show that the sequence $(T_n^{(p)})$ is consistent for θ, for $0 < p \leq \frac{1}{2}$.
(b) Compare the ARE of sequences of estimators $(T_n^{(p)})$ of θ for varying p – the answer will depend on the density function. How should p be chosen to maximize asymptotic efficiency?
(c) Evaluate your answer to 5(b) for the two examples of 4(a) and 4(b).