Stat 581 Homework 6: Due November 19, 2003

1.(a) TPE P.139, No. 5.23
(b) TPE P.140, No. 5.24
(These were the questions I meant to set in homework 5)

2. Continuing the same question as #1
(a) Show that the MLE, $\widehat{\theta_n}$ of θ is the integer closest to $\overline{X_n}$, and that this MLE is unbiased.
(b) Show that if $n \geq 15.5\sigma^2$, them $P(\widehat{\theta_n} = \theta) \geq 0.95$
(c) Show that $\widehat{\theta_n}$ is consistent.
(d) Show that $\text{var}(\widehat{\theta_n}) \;=\; 2\sum_1^\infty (2j-1)\;(1-\Phi((j-\frac{1}{2})n^{\frac{1}{2}}/\sigma)$.
(This can be used to show that the variance (like the bound) decays exponentially in n: see TPE P.140)

3. Let $(X_i, Y_i), i = 1, ..., n$ be i.i.d. bivariate normal, with $\mathrm{E}(X_i) = \mathrm{E}(Y_i) = 0$, $\text{var}(X_i) = \text{var}(Y_i) = 1$ and $\text{Cov}(X_i, Y_i) = \rho$. Find the Cramer-Rao lower bound on the variance of unbiased estimators of ρ. Does the sample correlation r achieve this bound asymptotically? Why/why not?

4. Let $X_i,\ i, ...m$ be i.i.d exponential with mean μ and $Y_1, ..., Y_n$ be i.i.d exponential with mean λ. Suppose the parameter of interest is $\theta = \mu/\lambda$. Compare the information about θ available from the two samples, with that available when λ is known.

5. (a) Suppose $X_1, ..., X_n$ are i.i.d $N(\theta, 1)$, so that $I_1(\theta) = 1$. Let $T_n = \overline{X_n}$ if $|\overline{X_n}| > n^{-1/4}$ and $T_n = a\overline{X_n}$ if $|\overline{X_n}| \leq n^{-1/4}$.
Show that $n^{\frac{1}{2}}(T_n - \theta) \to_d N(0, V^2(\theta))$ where $V^2(\theta) = 1$ if $\theta \neq 0$, and $V^2(\theta) = a^2$ if $\theta = 0$.
Choosing $|a| < 1$, then gives $V^2(\theta) < 1/I_1(\theta)$ at $\theta = 0$. Why does this not violate the CRLB theory?

(b) Suppose that $X_1, ..., X_n$ are i.i.d., with each X_i being $N(\theta, 1)$ with probability $\frac{1}{2}$ and $N(-\theta, 1)$ with probability $\frac{1}{2}$. Evaluate the information about θ. What happens when $\theta = 0$? What does this suggest to you about how you should estimate θ?