Stat 581 Homework 5: Due November 12, 2003

1. Let $f(x;\theta)$ be a density on $a(\theta) \le x \le b(\theta)$, where (for (a)-(c)) θ takes values in a non-degenerate interval. and let $X_1, ..., X_n$ be a sample from this density.

(a) Suppose $b(\theta) \equiv b$. Show that $X_{(1)} = \min(X_i)$ is sufficient if and only if $f(x;\theta) = g(x)/h(\theta)$.

(b) Suppose $f(x;\theta) = g(x)/h(\theta)$. Show that if $a(\theta)$ and $b(\theta)$ are both increasing, or both decreasing, as functions of θ, then $(X_{(1)}, X_{(n)}) = (\min(X_i), \max(X_i))$ is minimal sufficient.

(c) Suppose $f(x;\theta) = g(x)/h(\theta)$. If $a(\theta)$ and $b(\theta)$ are both monotone, but one is increasing and the other decreasing, find a one-dimensional sufficient statistic.

(d) If the density f is that of a uniform $U(-\theta,\theta)$ r.v. $(\theta > 0)$, find a one-dimensional sufficient statistic.

(e) If the density f is $U(\theta, \theta+1)$, where θ is an integer, show that any observation is sufficient and exhibit a strongly consistent estimator of θ.

2. Let $X_1, ..., X_n$ be i.i.d. positive random variables with density $f(x;\theta) = \theta^{-1}\exp(-x/\theta)$ and cdf $F(x;\theta)$. Let $Y_1, ..., Y_n$ be i.i.d. with known density g and cdf G. Suppose the vector of X_i are independent of the vector of Y_i.

Suppose we observe $Z_i = \min(X_i, Y_i)$ and $\delta_i = I\{X_i \le Y_i\}$. (You may want to think of the X_i as survival times and the Y_i as censoring times.)

(a) Show that the pairs (Z_i, δ_i) are i.i.d. with density

$$h(z,\delta) \quad = \quad (f(z;\theta)(1-G(z)))^{\delta}(g(z)(1-F(z;\theta))^{1-\delta} \qquad \delta = 0,1;\ 0 < z < \infty$$

(b) Find the Fisher information $I(\theta)$ based on observing (Z_i, δ_i), $i = 1, ..., n$ amd the resulting lower bound for the variance of unbiased estimators of θ.

3. Let X have density $\theta f_1(x) + (1-\theta) f_2(x)$ where f_1 and f_2 are known densities not depending on θ.

(a) Compute the information inequality lower bound on the variance of unbiased estimators of θ based on a sample of n observations.

(b) Show that the bound reduces to the bound from a sample from Bernoulli $Bin(1,\theta)$ r.vs when the suports of f_1 and f_2 do not overlap.

(c) More generally, show that the bound is bounded above by the value in (b). (Hint: See JAW Ch 3., P. 3.10.)