Stat 581 Homework 4: Due October 29, 2003

You may quote standard results and theorems, but should be specific about which one(s) you are citing, and why. This includes any theorems on my P. 3.4, even if we didn't get to them yet.

1. Based on Ferguson. P.24 # 4, and JAW Problems 3 #1
(a) Ferg. P.24 #4: Show that the SLLN fails, and hence that $\widehat{I_n}$ is likely not a good estimator of I.

(b) Generalize the integral I of (a) to

$$I_\alpha = \int_1^\infty x^{-\alpha} \sin(2\pi x) dx$$

Construct the corresponding estimator $\widehat{I_{n,\alpha}}$, using the same change-of-variable as in (a). For what values of α will the estimator $\widehat{I_{n,\alpha}}$ converge to I_α a.s.
Hint: bound $g(y)\sin(2\pi/y)$ by $g(y)$.

(c) For what values of α will $n^{\frac{1}{2}}(\widehat{I_{n,\alpha}} - I_\alpha)$ converge in distribution, and to what? (Same hint.)

2. Let $X_1, ..., X_n$ be i.i.d. with continuous density f. Let

$$f_n(x) = \frac{F_n(x + b_n) - F_n(x - b_n)}{2b_n}$$

where F_n is the empirical distribution function of $X_1, ..., X_n$. Thus f_n can be thought of as an empirical density estimate.

(a) Show that $\mathrm{E}(f_n(x)) \to f(x)$ if $b_n \to 0$.

(b) Show that $\mathrm{var}(f_n(x)) \to 0$ if $b_n \to 0$ and $nb_n \to \infty$.

(c) Show that (under suitable conditions) $(2nb_n)^{\frac{1}{2}}(f_n(x) - \mathrm{E}(f_n(x)) \to_d N(0, f(x))$

(d) Find conditions under which $\mathrm{E}(f_n(x))$ can be replaced by $f(x)$ in (c).

(e) Find the limiting distribution of $(f_n(x))^{\frac{1}{2}}$ suitably normalized.
Note: $(f_n(x)^{\frac{1}{2}} - f(x)^{\frac{1}{2}})$ is called a *rootogram*.

3. Let X_i be i.i.d. with mean μ variance σ^2 and finite fourth central moment $\mathrm{E}(X_i - \mu)^4 = \mu_4 = (3 + K)\sigma^4 < \infty$. Let $\overline{X_n} \equiv n^{-1}\sum_{i=1}^n X_i$.

(a) Let $Y_i \equiv (X_i - \mu)^2$, $\overline{Y_n} \equiv n^{-1}\sum_{i=1}^n Y_i$.
Show that $n^{\frac{1}{2}}(\overline{Y_n} - \sigma^2) \to_d N(0, (2 + K)\sigma^4)$.

(b) Let $S^2 \equiv (n-1)^{-1}\sum_{1=1}^n (X_i - \overline{X_n})^2$. Show that $n^{\frac{1}{2}}(S^2 - \sigma^2) \to_d N(0, (2 + K)\sigma^4)$.

(c) Show $S^2 \to_{a.s.} \sigma^2$.

(d) Show that $n^{\frac{1}{2}}(\overline{X_n} - \mu)/S \to_d N(0, 1)$, so that validity of the t-test is robust against departures from Normality of the X_i.

(e) Suppose $\sigma = \sigma_0$ is tested against the alternative $\sigma > \sigma_0$ by rejecting the null hypothesis if $((n-1)S^2 > c_n\sigma_0^2$ where $c_n = G^{-1}(1-\alpha)$ where G is the cdf of a χ^2_{n-1}. By considering the case when X_i are $N(\mu, \sigma^2)$ show that $\sqrt{n/2}(c_n(n-1)^{-1} - 1) \to \Phi^{-1}(1-\alpha)$.

(f) Show that, when $\sigma = \sigma_0$, in general the probability of rejecting under the test (e) converges to $1 - \Phi((1 + K/2)^{-\frac{1}{2}}\Phi^{-1}(1-\alpha))$, so that this test is not robust against departures from Normality which give $K \neq 0$.

4. Let $(X_i, Y_i)'$, $i = 1, ..., n$ be i.i.d. bivariate r.v.s with mean $(0,0)'$ and $\text{var}(X_i) = \text{var}(Y_i) = 1$, $\text{Cov}(X_i, Y_i) = \rho$.
Let $S_{ZW} = n^{-1}\sum_{i=1}^{n} Z_i W_i$, and $R = S_{XY}/\sqrt{S_{XX}S_{YY}}$. We investigate R as an estimator of ρ. Assume $\text{E}(X^4) < \infty$, $\text{E}(Y^4) < \infty$.

(a) Show that

$$\sqrt{n}(\overline{XY} - \rho, \overline{X^2} - 1, \overline{Y^2} - 1)' \quad \to_d \quad (Z_1, Z_2, Z_3)' \ \sim \ N_3(0, \Sigma)$$

where $\Sigma_{11} = \text{E}(X^2Y^2) - \rho^2$, $\Sigma_{12} = \text{E}(X^3Y) - \rho$, $\Sigma_{22} = \text{E}(X^4) - 1$, $\Sigma_{23} = \text{E}(X^2Y^2) - 1$.

(b) Show that $n^{\frac{1}{2}}(R - \rho) \to_d (Z_1 - \rho(Z_2 + Z_3)/2)$.

(c) Show that if X_i and Y_i are independent then $n^{\frac{1}{2}}(R - \rho) \ \to_d N(0,1)$ regardless of the underlying distribution (subject to the inital assumptions $\text{E}(X^4) < \infty$ etc.).

(d) Show that if (X_i, Y_i) is bivariate Normal, $\Sigma_{11} = 1 + \rho^2$, $\Sigma_{12} = 2\rho$, $\Sigma_{22} = 2)$, $\Sigma_{23} = 2\rho^2$,
and hence that $n^{\frac{1}{2}}(R - \rho) \ \to_d N(0, (1 - \rho^2)^2)$.

(e) Let $g(x) \ = \ \frac{1}{2}\log((1 + x)/(1 - x))$, $V = g(R)$, $\xi = g(\rho)$.
Show that $n^{\frac{1}{2}}(V - \xi) \to_d N(0,1)$.

5. Let $X_1, ..., X_n$ be i.i.d. with mean μ and k th central moment $\mu_k = \text{E}((X_1 - \mu)^k)$, and assume $\mu_{2k} < \infty$.
Define $B_k = n^{-1}\sum_{i-1}^{n}(X_i - \mu)^k$, and $M_k == n^{-1}\sum_{i-1}^{n}(X_i - \overline{X_n})^k$, where $\overline{X_n} = n^{-1}\sum_{i-1}^{n} X_i$.

(a) Show $B_k \to \mu_k$ a.s.

(b) Show that $n^{\frac{1}{2}}((B_1, ..., B_k)' - (0, \mu_2, ..., \mu_k)'$ is asymptotically (jointly) normal with mean $\mathbf{0}$ and variance $\mathbf{V}$ where $V_{ij} = \mu_{i+j} - \mu_i\mu_j$.

(c) Show that $M_k = \sum_{j=0}^{k}\frac{k!}{j!(k-j)!}(-1)^{k-j}B_jB_1^{k-j}$, and deduce that $M_k \to \mu_k$ a.s.

(d) Show that

$$n^{\frac{1}{2}}((M_i - \mu_i) \ - \ (B_i - \mu_i - i\mu_{i-1}B_1))$$

converges in probability to 0 as $n \to \infty$.

(e) Deduce that $n^{\frac{1}{2}}(M_2 - \mu_2, ..., M_k - \mu_k)'$ is asymptotically Normal $N_{k-1}(\mathbf{0}, \Sigma)$ where
$\sigma_{ij} = \mu_{i+j+2} - (i+1)\mu_i\mu_{j+2} - (j+1)\mu_{i+2}\mu_j + (i+1)(j+1)\mu_{i+1}\mu_{j+1}\mu_2$.

(f) Determine the asymptotic distribution of $G_2 \equiv (M_4/M_2^2 \ - \ 3)$.