**10. Gamma distributions: LM 4.6**

**10.1 The Gamma function** $\Gamma(\alpha)$**.** (LM P.329)

(i) Definition: $\Gamma(\alpha) \ = \ \int_0^\infty e^{-y} y^{\alpha-1} dy$

(ii) Integrating by parts:

$$\begin{aligned}\Gamma(\alpha) &= [-e^{-y}y^{\alpha-1}]_0^\infty \ + \ \int_0^\infty e^{-y}(\alpha-1)y^{\alpha-2}dy \\ &= (\alpha-1)\int_0^\infty e^{-y}y^{\alpha-2}dy \ = \ (\alpha-1)\Gamma(\alpha-1)\end{aligned}$$

Note $\Gamma(n) = (n-1)\Gamma(n-2) = .... = (n-1)!\Gamma(1) = (n-1)!$ since $\Gamma(1) = \int_0^\infty \exp(-y)dy \ = \ 1.$

**10.2 The Gamma density** $G(\alpha, \lambda)$. $\alpha > 0, \ \lambda > 0$. (LM P. 329)

(i) Definition: $f_Y(y) = \lambda^\alpha y^{\alpha-1}\exp(-\lambda y)/\Gamma(\alpha)$ for $0 < y < \infty$ and 0 otherwise.

Note 1: $\int_0^\infty f_Y(y)dy \ = \ 1$. (Substitute $v = \lambda y$).

Note 2: if $Y \sim G(\alpha, \lambda)$, $\lambda Y \ \sim \ G(\alpha, 1)$: $1/\lambda$ is a scale parameter.

(ii)

$$\begin{aligned}\mathrm{E}(Y^k) &= \int_0^\infty y^k \ f_Y(y) \ dy \ = \ (\Gamma(\alpha))^{-1}\int_0^\infty \lambda^\alpha y^{k+\alpha-1}\exp(-\lambda y) \ dy \\ &= (\Gamma(\alpha))^{-1}\int_0^\infty \lambda^{-k+1}v^{k+\alpha-1}\exp(-v) \ dv/\lambda \ = \ \frac{\Gamma(k+\alpha)}{\lambda^k\Gamma(\alpha)}\end{aligned}$$

(iii) The mean and variance of a Gamma random variable (LM. P.330) (or use Mgf below)

$$\mathrm{E}(Y) \ = \ \Gamma(1+\alpha)/\lambda\Gamma(\alpha) \ = \ \alpha/\lambda, \qquad \mathrm{E}(Y^2) \ = \ \Gamma(2+\alpha)/\lambda^2\Gamma(\alpha) \ = \ \alpha(\alpha+1)/\lambda^2.$$

Hence $\mathrm{var}(Y) \ = \ \mathrm{E}(Y^2) - (\mathrm{E}(Y))^2 \ = \ \alpha(\alpha+1)/\lambda^2 \ - \ (\alpha/\lambda)^2 \ = \ \alpha/\lambda^2$.

**10.3 The Mgf of** $G(\alpha, \lambda)$

(i)

$$\mathrm{E}(e^{tX}) \ = \ (\lambda^\alpha/\Gamma(\alpha))\int_0^\infty x^{\alpha-1}\exp(-(\lambda-t)x) \ dx \ = \ (\lambda/(\lambda-t))^\alpha$$

(ii) From 9.3, the Mgf of an exponential $\mathcal{E}(\lambda)$ r.v. is $\lambda/(\lambda-t)$.

Suppose $Y_1, ..., Y_n$ are i.i.d. exponential $\mathcal{E}(\lambda)$.

Then the Mgf of $\sum_{i=1}^n Y_i$ is $\prod_{i=1}^n \lambda/(\lambda-t) \ = \ (\lambda/(\lambda-t))^n$.

But this is the Mgf of a $G(n, \lambda)$ random variable.

(iii) Hence, by uniqueness of Mgf's, the distribution of $\sum_{i=1}^n Y_i$ is $G(n, \lambda)$– summing independent exponential r.v.s with the same $\lambda$ gives a Gamma r.v..

**10.4 Summing and scaling Gamma distributions. LM P.330-332**

(i) $X_1 \ \sim G(\alpha_1, \lambda)$, $X_2 \ \sim \ G(\alpha_2, \lambda)$, $X_1$ and $X_2$ independent.

Then Mgf of $X_1 + X_2$ is $(\lambda/(\lambda-t))^{\alpha_1}.(\lambda/(\lambda-t))^{\alpha_2} \ = \ (\lambda/(\lambda-t))^{\alpha_1+\alpha_2}$, which is Mgf of $G(\alpha_1+\alpha_2, \lambda)$.

Hence, by uniqueness of Mgf, $X_1 + X_2 \ \sim \ G(\alpha_1+\alpha_2, \lambda)$.

Note this works for Gamma r.v.s with different $\alpha 1, \alpha_2, ...$, but they must have the same $\lambda$.

(ii) We know that if $X \ \sim \ \mathcal{E}(\lambda)$ then $kX \ \sim \ \mathcal{E}(\lambda/k)$.

The same works for Gamma r.v.s; we can use the Mgf to show this also: If $X \ \sim \ G(\alpha, \lambda)$

$$\mathrm{E}(\exp((kX)t)) \ = \ \mathrm{E}(\exp((kt)X)) \ = \ M_X(kt) \ = \ (\lambda/(\lambda-kt))^\alpha \ = \ ((\lambda/k)/((\lambda/k)-t))^\alpha$$

So, by uniqueness of Mgf, $kX \ \sim \ G(\alpha, \ \lambda/k)$

**11. Chi-squared distributions: Sums of squares of independent Normal r.vs; LM P474**

**11.1 Definition of $\chi^2_m$ distribution**

If $Z_1, ....., Z_m$ are independent standard Normal, $N(0,1)$, random variables, then $Y = \sum_{i=1}^m Z_i^2$ has a chi-squared distribution, $\chi^2_m$, with $m$ *degrees of freedom*.

**11.2 The Mgf of a $\chi^2_1$ distribution**

(i) First consider the Mgf of $Z^2$, where $Z \sim N(0,1)$; a $\chi^2_1$ distribution.

$$\begin{aligned} m_{Z^2}(t) &= \mathrm{E}(\exp(tZ^2)) = \int e^{tz^2} f_Z(z)\, dz \\ &= \int_{-\infty}^{\infty} (1/\sqrt{2\pi}) \exp(-z^2/2 + tz^2)\, dz \\ &= \int_{-\infty}^{\infty} (1/\sqrt{2\pi}) \exp(-z^2(1-2t)/2)\, dz \\ &= (1-2t)^{-1/2} \quad \text{substituting } w = \sqrt{1-2t}z \end{aligned}$$

(ii) So the Mgf of $\chi^2_1$ is $(1-2t)^{-1/2} = ((1/2)/((1/2)-t))^{1/2}$.
But this is the Mgf of a $G(1/2,\ 1/2)$, so by uniqueness of Mgf a $\chi^2_1$ distribution is a $G(1/2,\ 1/2)$ distribution.

**11.3 The relationship of exponentials, chi-squared and Gamma dsns**

(i) So now a $\chi^2_m$ r.v. is $Z_1^2 + Z_2^2 + ... + Z_m^2$, where $Z_i$ are independent $N(0,1)$.
So now the $\chi^2_m$ distribution has Mgf $((1-2t)^{-1/2})^m = (1-2t)^{-m/2} = ((1/2)/((1/2)-t))^{m/2}$.
But this is the Mgf of a $G(m/2,\ 1/2)$, so by uniqueness of Mgf a $\chi^2_m$ distribution is a $G(m/2,\ 1/2)$ distribution.
(ii) A $\chi^2_2$ distribution is has Mgf $(1-2t)^{-1} - (1/2)/((1/2)-t)$.
But this is Mgf of exponential $\mathcal{E}(1/2)$ or Gamma $G(1,\ 1/2)$.
So, by uniqueness of Mgf, a $\chi^2_2$ distribution is $G(1, 1/2) \equiv \mathcal{E}(1/2)$.
Or, if $X \sim N(0,1)$ and $Y \sim N(0,1)$, with $X$ and $Y$ independent, then $(X^2+Y^2) \sim \mathcal{E}(1/2)$.

**11.4 $\sum(X_i^2)$ for $X_i$ i.i.d $N(0,\sigma^2)$**

(i) Normals scale, exponentials scale, Gammas scale, and so do chi-squareds.
(ii) $Z_i \equiv X_i/\sigma \sim N(0,1)$. $\sum_{i=1}^m Z_i^2 \sim \chi^2_m$, or $\sum_{i=1}^m X_i^2 \sim \sigma^2\chi^2_m$.
(iii) $\sigma^2\chi^2_m \equiv \sigma^2 G(m/2, 1/2) \equiv G(m/2, 1/(2\sigma^2))$.
(iv) From 10.2, If $W \sim G(m/2, 1/(2\sigma^2))$ then $W$ has expectation $(m/2)/(1/2\sigma^2) = m\sigma^2$ and variance $(m/2)/(1/2\sigma^2)^2 = 2m\sigma^4$.
(v) Recall $T = (1/m)\sum_{i=1}^m X_i^2$ was our MoM estimator of $\sigma^2$ (see 4.3).
So now $T = W/m$, $\mathrm{E}(T) = \mathrm{E}(W)/m = \sigma^2$, so $T$ is an unbiased estimator of $\sigma^2$ – which we knew already.
Also, (this is new), $\mathrm{var}(T) = \mathrm{var}(W)/m^2 = 2\sigma^4/n$; we now have a formula for the variance (or mse) of this estimator.