All the details of $X_1, ..., X_n \sim U(0, \theta)$.

Examples to be worked: Wed Jan 20, 2010

1. The MoM estimator

(a) If $X \sim U(0, \theta)$, show $\mathrm{E}(X) = \theta/2$ and $\mathrm{var}(X) = \theta^2/12$.

(b) If $X_1, ..., X_n$ are i.i.d. $U(0, \theta)$, show the MoM estimator of θ is $2\overline{X_n} = (2/n)\sum_{i=1}^{n} X_i$.

(c) Show the MoM estimator of θ is unbiased.

(d) Show the MoM estimator of θ has mean square error (mse) $\theta^2/3n$.

2. The maximum, W, of an n-sample of $U(0, \theta)$

(a) Show the pdf of W is $f_W(w) = nw^{n-1}/\theta^n$ on $0 \le w \le \theta$, and 0 otherwise.

(b) Show $\mathrm{E}(W) = n\theta/(n + 1)$, $\mathrm{E}(W^2) = n\theta^2/(n + 2)$, $\mathrm{var}(W) = n\theta^2/(n + 1)^2(n + 2)$.

(c) Show that W is a negatively biased estimator of θ, but that $T = (n + 1)W/n$ is an unbiased estimator.

(d) Show that the mse of T as an estimator of θ is $\theta^2/n(n + 2)$.

3. Other multiples of W or T

(a) Consider kT where k is a constant. Write down the mean and variance of kT.

(Note $kT = k(n + 1)W/n$, but it is less messy to work with kT directly.)

(b) Show that the bias of kT as an estimator of θ is $(k - 1)\theta$.

(c) Show that the mse of kT as an estimator of θ is

$$\theta^2(k^2/n(n + 2) \; + \; (k - 1)^2) \; = \; \theta^2 \; (k^2(n + 1)^2 \; - \; 2kn(n + 2) + n(n + 2))/n(n + 2)$$

(d) Show that the mse is minimized w.r.t. k by $k = n(n + 2)/(n + 1)^2$.

That is, the estimator $n(n + 2)T/(n + 1)^2 = (n + 2)W/(n + 1)$ has smallest mse.

4. If you find it hard, at first, to work with general n, you may work with, for example, $n = 5$ (as in the homework).

4.1 Then the MoM estimator $(2/5)(X_1 + ... + X_5)$ has mean θ and variance (or mse) $\theta^2/15$.

4.2 (b) $\mathrm{E}(W) = 5\theta/6$, $\mathrm{E}(W^2) = 5\theta^2/7$, $\mathrm{var}(W) = 5\theta^2/252$.

4.2 (c) $(\mathrm{E}(W) \; - \; \theta \; = \; ((5/6) - 1)\theta \; = \; -\theta/6 \; < \; 0$.

$T = 6W/5$ is unbiased: $\mathrm{E}(T) = (6/5).(5\theta/6) = \theta$, for any θ.

4.2 (d) T has variance (or mse) $(36/25).\mathrm{var}(W) = \theta^2/35$.

4.3 (a) $\mathrm{var}(kT) = k\theta$ (T unbiased). $\mathrm{var}(kT) = k^2\theta^2/35$.

4.3 (b) $\mathrm{E}(kT) - \theta = k\theta - \theta = (k - 1)\theta$.

4.3 (c) mse of kT is $\mathrm{var}(kT) + (b_{kT}(\theta))^2$ which reduces to $(\theta^2/35)(36k^2 \; - \; 70k + 70)$

4.3 (d) This is minimized by $k = 35/36$. So the estimator with smallest mse is $35T/36$ or $(35/36).(6W/5) = 7W/6$.