STAT 341 - Elizabeth Thompson

Homework 8 Solutions

LM 5.8.1, 5.8.4, 5.8.5, 5.8.6, 5.8.7, 5.8.8

(5.8.1)

By definition 5.8.1,

$$g_\Theta(\theta|X=k) = \frac{p_X(k|\theta)f_\Theta(\theta)}{\int_{-\infty}^{\infty} p_X(k|\theta)f_\Theta(\theta)d\theta}$$

$$\begin{aligned} p_X(k|\theta)f_\Theta(\theta) &= \left[(1-\theta)^k\theta\right]\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\theta^{r-1}(1-\theta)^{s-1} \\ &= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\theta^r(1-\theta)^{s+k-1} \end{aligned}$$

Note that the constant term $\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}$ will cancel out with the denominator. Then, $\theta^r(1-\theta)^{s+k-1}$ is the kernel of a beta distribution with parameters $(r+1, s+k)$. Thus,

$$\int_0^1 \theta^r(1-\theta)^{s+k-1}d\theta = \frac{\Gamma(r+1)\Gamma(s+k)}{\Gamma(r+s+k+1)}$$

and

$$g_\Theta(\theta|X=k) == \frac{\Gamma(r+s+k+1)}{\Gamma(r+1)\Gamma(s+k)}\theta^r(1-\theta)^{s+k-1}$$

which is a beta distribution with parameters $(r+1, s+k)$.

(5.8.4)

By Theorem 5.8.1, the squared-error loss Bayes estimate is the mean of the posterior.

$$p_X(k|\theta)f_\Theta(\theta) =_n C_k\theta^k(1-\theta)^{n-k} \times 1$$

$\theta^k(1-\theta)^{n-k}$ is the kernel of a beta distribution with parameters $(k+1, n-k+1)$. The mean of which is

$$\frac{r}{s+r} = \frac{k+1}{k+1+n-k+1} = \frac{k+1}{n+2}$$

Note that we can write this as a weighted average of the MLE for θ and the mean of the prior distribution.

$$\frac{k+1}{n+2} = \frac{n}{n+2}\left(\frac{k}{n}\right) + \frac{2}{n+2}\left(\frac{1}{2}\right)$$

(5.8.5)

In each case the estimator is biased, since the mean of the estimator is a weighted average of the unbiased maximum likelihood estimator and a non-zero constant. However, in each case the weighting on the maximum likelihood estimator tends to 1 as n tends to ∞, so these estimators are asymptotically unbiased.

(5.8.6)

$$\begin{aligned} f_Y(y|\theta)f_\Theta(\theta) &= \frac{\theta^r}{\Gamma(r)} y^{r-1} e^{-\theta y} \frac{\mu^s}{\Gamma(s)} \theta^{s-1} e^{-\mu\theta} \\ &= \frac{\mu^s y^{r-1}}{\Gamma(r)\Gamma(s)} \theta^{r+s-1} e^{-(y+\mu)\theta} \end{aligned}$$

$\theta^{r+s-1}e^{-(y+\mu)\theta}$ is the kernel of a gamma distribution with parameters $(r+s, y+\mu)$ which is the posterior distribution.

(5.8.7)

Since the sum of gamma random variables is gamma, W is gamma with parameters (nr, λ). The posterior will then also be gamma with parameters $(nr+s, \sum y + \mu)$. (See 5.8.6).

(5.8.8)

By Theorem 5.8.1, the squared-error loss Bayes estimate is the mean of the posterior. From 5.8.7 we know the posterior is Gamma$(nr+s, \sum y + \mu)$ with mean,

$$\frac{nr+s}{\sum y + \mu}$$