STAT 341 - Elizabeth Thompson

Homework 7 Solutions

LM 5.6.1, 5.6.2, 5.6.5, 5.5.2, 5.5.3, 5.5.4

(5.6.1)

$$\prod_{i=1}^{n} p_X(x_i;p) = \prod_{i=1}^{n}(1-p)^{x_i-1}p = p^n(1-p)^{\sum_{i=1}^n x_i - n}$$

Let $g(\sum_{i=1}^n x_i; p) = p^n(1-p)^{\sum_{i=1}^n x_i - n}$ and $b(x_1, \ldots, x_n) = 1$.
Then by Theorem 5.6.1, the statistic $\sum_{i=1}^n X_i$ is sufficient.

(5.6.2)

(a) Let $1_{(y_i \geq \theta)}$ denote the indicator function, ie.

$$1_{(y_i \geq \theta)} = \begin{cases} 1 & \text{if } y_i \geq \theta \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} e^{-(y_i-\theta)} 1_{(y_i \geq \theta)} \\ &= e^{-\sum y_i} e^{n\theta} 1_{(y_{min} \geq \theta)} \end{aligned}$$

Since $y_i \geq \theta$ for $i = 1, \ldots, n$ if and only if $y_{min} \geq \theta$,

$$\prod_{i=1}^{n} 1_{(y_i \geq \theta)} = 1_{(y_{min} \geq \theta)}$$

Let $g(y_{min}; \theta) = e^{n\theta} 1_{(y_{min} \geq \theta)}$ and $b(y_1, \ldots, y_n) = e^{-\sum y_i}$.
Then by Theorem 5.6.1, the statistic Y_{min} is sufficient.

(b) For Y_{max} to be sufficient, the likelihood function given y_{max} must be independent of θ. However, regardless the value of y_{max}, the likelihood function, shown above, still depends on θ. Thus Y_{max} is not sufficient.

(5.6.5)

$$
\begin{aligned}
L(\sigma^2) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{y_i^2}{2\sigma^2}} \\
&= \left[(\sigma^2)^{-\frac{1}{2}} e^{-\frac{\sum y_i^2}{2\sigma^2}} \right] [2\pi]^{-\frac{1}{2}} \\
&= g(\sum y_i^2; \sigma^2) b(y_1, \ldots, y_n)
\end{aligned}
$$

Thus by Theorem 5.6.1 $\sum_{i=1}^{n} y_i^2$ is sufficient.

(5.5.2)

$$
\begin{aligned}
\ln f_Y(Y;\theta) &= -\ln\theta - Y/\theta \\
\frac{\partial f_Y(Y;\theta)}{\partial\theta} &= -\frac{1}{\theta} + \frac{Y}{\theta^2} \\
\frac{\partial^2 f_Y(Y;\theta)}{\partial\theta^2} &= \frac{1}{\theta^2} - \frac{2Y}{\theta^3} \\
\mathrm{E}\left[\frac{\partial^2 f_Y(Y;\theta)}{\partial\theta^2}\right] &= \frac{1}{\theta^2} - \frac{2\theta}{\theta^3} = -\frac{1}{\theta^2} \\
\text{CRLB} &= \frac{\theta^2}{n}
\end{aligned}
$$

$$
\mathrm{Var}(\hat{\theta}) = \mathrm{Var}(\bar{Y}) = \frac{\theta^2}{n}
$$

Therefore $\hat{\theta}$ achieves the CRLB and is a best estimator.

(5.5.3)

$$
\begin{aligned}
\ln f_X(X;\lambda) &= -\lambda + X\ln\lambda - \ln X! \\
\frac{\partial f_X(X;\lambda)}{\partial\lambda} &= -1 + X/\lambda \\
\frac{\partial^2 f_X(X;\lambda)}{\partial\lambda^2} &= -X/\lambda^2 \\
\mathrm{E}\left[\frac{\partial^2 f_X(X;\lambda)}{\partial\lambda^2}\right] &= -\lambda/\lambda^2 = -1/\lambda \\
\text{CRLB} &= \lambda/n
\end{aligned}
$$

$$
\mathrm{Var}(\hat{\lambda}) = \mathrm{Var}(\bar{X}) = \lambda/n
$$

Therefore $\hat{\lambda}$ achieves the CRLB and is an efficient estimator.

(5.5.4)

$$\begin{aligned}
\ln f_Y(Y;\mu) &= -\ln\sqrt{2\pi\sigma} - \frac{(Y-\mu)^2}{2\sigma^2} \\
\frac{\partial f_Y(Y;\mu)}{\partial\mu} &= \frac{Y-\mu}{\sigma^2} \\
\frac{\partial^2 f_Y(Y;\mu)}{\partial\mu^2} &= -1/\sigma^2 \\
\mathrm{E}\left[\frac{\partial^2 f_Y(Y;\mu)}{\partial\mu^2}\right] &= -1/\sigma^2 \\
\text{CRLB} &= \frac{\sigma^2}{n}
\end{aligned}$$

$$\mathrm{Var}(\hat{\mu}) = \mathrm{Var}(\bar{Y}) = \frac{\sigma^2}{n}$$

Therefore $\hat{\mu}$ achieves the CRLB and is an efficient estimator.