STAT 341 - Elizabeth Thompson

Homework 6 Solutions

LM 5.2.4, 5.2.6, 5.2.9, 5.2.11, 5.2.12, 5.2.14

(5.2.4)

$$\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} \frac{\theta^{2k_i} e^{-\theta^2}}{k_i!} = \frac{\theta^{2\sum k_i} e^{-n\theta^2}}{\prod k_i!} \\
\ln L(\theta) &= \left(2\sum_{i=1}^{n} k_i\right) \ln\theta - n\theta^2 + \ln\prod_{i=1}^{n} k_i! \\
\frac{d\ln L(\theta)}{d\theta} &= \frac{2\sum k_i}{\theta} - 2n\theta = \frac{2\sum k_i - 2n\theta^2}{\theta} = 0 \\
\hat{\theta}_{ML} &= \sqrt{\frac{\sum k_i}{n}}
\end{aligned}$$

(5.2.6)

$$\begin{aligned}
L(\theta) &= \prod_{i=1}^{4} \frac{\theta e^{-\theta\sqrt{y_i}}}{2\sqrt{y_i}} = \frac{\theta^4 e^{-\theta\sum\sqrt{y_i}}}{16\prod\sqrt{y_i}} \\
\ln L(\theta) &= 4\ln\theta - \ln\left(16\prod_{i=1}^{4}\sqrt{y_i}\right) - \theta\sum_{i=1}^{4}\sqrt{y_i} \\
\frac{d\ln L(\theta)}{d\theta} &= \frac{4}{\theta} - \sum_{i=1}^{4}\sqrt{y_i} = 0 \\
\hat{\theta}_{ML} &= \frac{4}{\sum\sqrt{y_i}} = \frac{4}{8.766} = 0.456
\end{aligned}$$

(5.2.9)

(a) $L(\theta) = \left(\frac{1}{\theta}\right)^n$ for $0 \le y_1, y_2, \ldots, y_n \le \theta$ and 0 otherwise. This function is decreasing in θ. Therefore, to maximize the likelihood we need to choose the smallest value of θ that satisfies the constraint $0 \le y_1, y_2, \ldots, y_n \le \theta$. Thus, $\hat{\theta}_{ML} = y_{max}$ maximizes the likelihood. For these data this is 14.2.

(b) $L(\theta) = \left(\frac{1}{\theta_2 - \theta_1}\right)^n$ for $\theta_1 \le y_1, y_2, \ldots, y_n \le \theta_2$ and 0 otherwise. Similar to above, $\hat{\theta}_{1_{ML}} = y_{min} = 1.8, \hat{\theta}_{2_{ML}} = y_{max} = 14.2$ maximizes the likelihood.

(5.2.11) $L(\theta) = \prod_{i=1}^{n} 2y_i\theta^2 = 2^n \left(\prod y_i\right)\theta^{2n}$, if $0 \leq y_1, y_2, \ldots, y_n \leq 1/\theta$, and 0 otherwise. To maximize $L(\theta)$ we must maximize θ. Since each $y_i \leq 1/\theta$, then $\theta \leq 1/y_i$ for $1 \leq i \leq n$. The maximum value for θ under these constraints is the minimum of the $1/y_i$ or $\hat{\theta}_{ML} = 1/y_{max}$.

(5.2.12)

$$\begin{aligned}
L(\theta) &= \prod_{i=1}^{25} \theta k^{\theta}\left(\frac{1}{y_i}\right)^{\theta+1} = \theta^{25}k^{25\theta}\left(\frac{1}{\prod y_i}\right)^{\theta+1} \\
\ln L(\theta) &= 25\ln\theta + 25\theta\ln k - (\theta+1)\sum_{i=1}^{25}\ln y_i \\
\frac{d\ln L(\theta)}{d\theta} &= \frac{25}{\theta} + 25\ln k - \sum_{i=1}^{25}\ln y_i = 0 \\
\hat{\theta}_{ML} &= \frac{25}{-25\ln k + \sum \ln y_i}
\end{aligned}$$

(5.2.14) Let $\theta = \sigma^2$.

$$\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{1}{2}\frac{(y_i-\mu)^2}{\theta}} \\
&= 2\pi^{-n/2}\theta^{-n/2}e^{-\frac{1}{2\theta}\sum(y_i-\mu)^2} \\
\ln L(\theta) &= -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\theta - \frac{1}{2\theta}\sum_{i=1}^{n}(y_i-\mu)^2 \\
\frac{d\ln L(\theta)}{d\theta} &= -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n}(y_i-\mu)^2 = 0 \\
\hat{\theta}_{ML} &= \hat{\sigma}^2_{ML} = \frac{1}{n}\sum_{i=1}^{n}(y_i-\mu)^2
\end{aligned}$$

This is the same as example 5.2.4 except it uses the known value of μ instead of the ML estimate, $\hat{\mu}_{ML} = \bar{y}$.