STAT 341 - Elizabeth Thompson

Homework 3 Solutions

5.2.16: 5.2.23: 5.4.2; 5.4.7; 5.4.10; 5.7.2

(5.2.16)

$$E[Y] = \int_0^1 y(\theta^2+\theta)y^{\theta-1}(1-y)dy = (\theta^2+\theta)\int_0^1 y^{\theta}(1-y)dy$$

$$= (\theta^2+\theta)\left(\frac{1}{\theta+1} - \frac{1}{\theta+2}\right) = \theta - \frac{\theta^2+\theta}{\theta+2}$$

$$= \frac{\theta^2+2\theta}{\theta+2} - \frac{\theta^2+\theta}{\theta+2} = \frac{\theta}{\theta+2}$$

Set $\frac{\theta}{\theta+2} = \bar{y} \Rightarrow \theta_e = \frac{2\bar{y}}{1-\bar{y}}$.

(5.2.23)

$$E[X] = \frac{r}{p} \quad V[x] = \frac{r(1-p)}{p^2}$$

$$E[X^2] = V[X] + E[X]^2 = \frac{r(1-p)+r^2}{p^2}$$

Set $r/p = \bar{x} \Rightarrow r = p\bar{x}$ and substitute into

$$\frac{r(1-p)+r^2}{p^2} = \frac{1}{n}\sum_{i=1}^n x_i^2$$

Solving for p,

$$p\bar{x} - p^2\bar{x} + p^2\bar{x}^2 - p^2\frac{1}{n}\sum_{i=1}^n x_i^2 = 0$$

$$p_e = \frac{\bar{x}}{\bar{x} + \frac{1}{n}\sum_{i=1}^n x_i^2 - \bar{x}^2}$$

$$r_e = p_e\bar{x} = \frac{\bar{x}^2}{\bar{x} + \frac{1}{n}\sum_{i=1}^n x_i^2 - \bar{x}^2}$$

(5.4.2)

(a) $n = 6$ and $\theta = 3$.

$$f_{\hat{\theta}}(u) = n\frac{1}{\theta}\left(\frac{u}{\theta}\right)^{n-1} = \frac{2}{243}u^5$$

$$P(|\hat{\theta}-3|<0.2)=\int_{2.8}^{3}\frac{2}{243}u^5du=\frac{2}{243}\frac{u^6}{6}|_{2.8}^{3}=.339$$

(b) $n=3$

$$f_{\hat{\theta}}(u)=\frac{1}{9}u^2$$

$$P(|\hat{\theta}-3|<0.2)=\int_{2.8}^{3}\frac{1}{9}u^2du=\frac{1}{9}\frac{u^3}{3}|_{2.8}^{3}=.187$$

(5.4.7)

$$E[Y]=\int_{\theta}^{\infty}ye^{-(y-\theta)}dy$$

Set $u=y-\theta$ then,

$$E[Y]=\int_{0}^{\infty}(u+\theta)e^{-u}du=1+\theta$$

$$E[\bar{Y}]=E[Y]=1+\theta\Rightarrow E[\bar{Y}-1]=\theta$$

(5.4.10)

$E[Y^2]=\int_0^\theta y^2\frac{1}{\theta}dy=\frac{\theta^2}{3}$, So $3Y^2$ is unbiased.

(5.7.2)

Since $\mu=0$, $E[Y_i^2]=\sigma^2$. By Chebyshev's inequality:

$$P(|S_n^2-\sigma^2|<\epsilon)>1-\frac{V[S_n^2]}{\epsilon^2}$$

$$V[S_n^2]=\frac{1}{n^2}\sum_{i=1}^{n}V[Y_i^2]=\frac{V[Y_i^2]}{n}$$

Note that since $Y_i\sim N(0,\sigma^2)\Rightarrow Y_i^2\sim\sigma^2\chi^2(1)$, hence $V[Y_i^2]=2\sigma^4$, although all that is necessary for this proof is the $V[Y_i^2]<\infty$.

$$P(|S_n^2-\sigma^2|<\epsilon)>1-\frac{V[Y_i^2]}{n\epsilon^2}$$

For any ϵ,δ, an n can be found that makes $\frac{V[Y_i^2]}{n\epsilon^2}<\delta$. Thus,

$$\lim_{n\to\infty}P(|S_n^2-\sigma^2|<\epsilon)=1$$

and S_n^2 is a consistent estimator of σ^2.