1. (*16 points: 4 each part*)

(a) $M_X(t) = \mathrm{E}(\exp(tX))$ and $M_{kX}(t) = \mathrm{E}(\exp(t(kX))) = \mathrm{E}(\exp((kt)X)) = M_X(kt)$.
$M_{X+Y}(t) = \mathrm{E}(\exp(t(X+Y))) = \mathrm{E}(\exp(tX).\exp(tY)) = \mathrm{E}(\exp(tX)).\mathrm{E}(\exp(tY))$, using independence of X and Y.

(b) mgf of Y_i is $(1/2)/((1/2)-t)$ from info sheet, so using (a)
mgf of $(1/2)(Y_1+Y_2+Y_3)$ is $((1/2)/((1/2)-(t/2)))^3 = (1/(1-t))^3$
mgf of W_i is $(1/(1-t))^{1.5}$ from info sheet, so usin (a), mgf of W_1+W_2 is $(1/(1-t))^3$
Also, from the info sheet, this is mgf of $G(3,1)$.
So, by uniqueness of mgf, $(1/2)(Y_1+Y_2+Y_3)$ and W_1+W_2 have same $G(3,1)$ distribution.

(c) Note $(X_1^2+X_2^2) = 3(Z_1^2+Z_2^2)$, where $Z_i \sim N(0,1)$, and from table Z_i^2 has mgf $(\frac{1}{2}/(\frac{1}{2}-t))^{1/2}$
So mgf of $X_1^2+X_2^2$ is $(\frac{1}{2}/(\frac{1}{2}-3t))^{2.(1/2)} = (1/6)/((1/6)-t)$.
But mgf of $3Y_1$ is also $(\frac{1}{2}/(\frac{1}{2}-3t)) = ((1/6)/((1/6)-t))$ which is mgf of $\mathcal{E}(1/6)$.
So, by uniqueness of mgf, $X_1^2+X_2^2$ and $3Y_1$ have the same $\mathcal{E}(1/6)$ distribution.

(d) As in (c) mgf of $(X_1^2+X_2^2+X_3^2)$ is $(\frac{1}{2}/(\frac{1}{2}-3t))^{3.(1/2)} = (1/6)/((1/6)-t)^{1.5}$.
And mgf of $6W_3$ is $(1/(1-6t))^{1.5} = ((1/6)/((1/6)-t))^{1.5}$ which is mgf of $G(1.5,1/6)$.
So, by uniqueness of mgf, $(X_1^2+X_2^2+X_3^2)$ and $6W_3$ have same $G(1.5,1/6)$ distribution.

2. (*20 points: 4 each part*)

Suppose that $x_1,...,x_n$ are the outcomes of n-sample $X_1,...,X_n$ which are i.i.d from the probability density function $f_X(x;\theta) = \theta x^{\theta-1}/2^\theta$ on $0 \le x \le 2$ (and $f_X(x;\theta) = 0$ otherwise), where $\theta > 0$.

(a) $\mathrm{E}(X_i) = \int_0^2 \theta x^\theta/2^\theta = \theta[x^{\theta+1}/(\theta+1)]_0^2/2^\theta = 2\theta/(\theta+1)$.

(b) Method of moments equation is $\overline{x_n} = \mathrm{E}(X_i) = 2\theta/(\theta+1)$, which gives $\theta = \overline{x_n}/(2-\overline{x_n})$.
So the MoM estimator of θ is $\overline{X_n}/(2-\overline{X_n})$.

(c)

$$L_n(\theta) = \prod_{i=1}^n f_X(x_i;\theta) = \prod_{i=1}^n \theta x_i^{\theta-1}/2^\theta = \theta^n(\prod_{i=1}^n x_i)^{\theta-1}/2^{n\theta}.$$

so by the factorization criterion $T = \prod_{i=1}^n X_i$ is a sufficient statistic for θ.

(d) Writing t for the value of $T = \prod_{i=1}^n X_i$, the log-likelihood is

$$\ell_n(\theta) = n \log\theta + (\theta-1)\log t - n\theta\log 2$$
$$\ell_n'(\theta) = n/\theta + \log t - n\log 2 = 0$$

gives estimator $1/(\log(2)-(1/n)\log(T))$ or $1/(\log 2 - (1/n).\sum_{i=1}^n \log(X_i))$.

(e) The MLE should be preferred.
The main reason is that the MLE is a function of the sufficient statistic, while the MoM estimator is not. The Rao=Blackwell Theorem tells we can always do better using only functions of sufficient statistics.
(For large samples, MLE's reliably have other good properties – asymptotically unbiased, consistent, small mse, etc.)