Problem 1. (12 + 2 points)

A random sample of size n is to be taken from a Uniform $[0,\theta]$ distribution. That is, $X_1, X_2, \ldots X_n$ are iid Uniform $[0,\theta]$. Consider the two estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ where

$$\hat{\theta}_1 = 2\overline{X} \quad \text{and} \quad \hat{\theta}_2 = \frac{n+1}{n} X_{max} .$$

Parts (a) and (b) relate to $\hat{\theta}_1$. Part (c) is optional and may be worked for bonus points. Parts (c) and (d) relate to $\hat{\theta}_2$.

(a) What is the bias of $\hat{\theta}_1$? Show and explain all of your work. Be sure to clearly specify your final answer.

(b) What is Var($\hat{\theta}_1$)? Show and explain all of your work. Be sure to clearly specify your final answer.

(c) (This problem part is OPTIONAL.)
The density of $\hat{\theta}_2$ is as follows. For 2 bonus points, derive this density. Your derivation must be clear and complete. [Hint: You likely did this for a recent homework problem.]

$$f_{\hat{\theta}}(y) = \frac{n^{n+1}}{(n+1)^n \theta^n} y^{(n-1)} \quad \text{for } 0 \le y \le \theta(\frac{n+1}{n})$$

$$= \quad 0 \qquad \qquad \text{elsewhere}$$

(d) Fact: $\hat{\theta}_2$ is an unbiased estimator of θ.
Derive Var($\hat{\theta}_2$). Your derivation likely will require several steps. Show and explain all of your work. Be sure to clearly specify your final answer. [Hint: Find Var(X_{max}).]

(e) What is the relative efficiency of $\hat{\theta}_1$ and $\hat{\theta}_2$? Show and explain all of your work. Give the definition of relative efficiency, then show how you found the values of each of the pieces necessary for its computation. Be sure to clearly specify your final answer.

(f) Are either of $\hat{\theta}_1$ and $\hat{\theta}_2$ consistent estimators for θ ? For this problem part, CHOOSE ONE of the two estimators and show whether or not it is consistent. Clearly specify which estimator you have chosen, then show and clearly explain all of your work. You should include a clear statement of how one determines whether or not an estimator is consistent.

Problem 2. (11 points)

Consider the following bivariate pdf.

$$f_{X,Y}(x,y) = x\, e^{-(x+xy)} \qquad \text{for } x > 0 \text{ and } y > 0$$

$$= 0 \qquad\qquad\qquad \text{elsewhere}$$

For all parts of this problem <u>show the derivations of your answers</u> even if you know by looking at the problem what the answers should be. Read both (a) and (b) before working on either.

(a) Find the **marginal CDF** for Y, $F_Y(y)$. Show and explain clearly how you derived your answer. Be sure to define explicitly the relevant range(s) for x and y, and to list all cases.

(b) Derive the **marginal PDF** for Y, $F_Y(x)$. Show and explain clearly how you derived your answer. Be sure to define explicitly the relevant range(s) for x and y, and to list all cases.

(c) Find the **conditional pdf** for $(X \mid Y=3) = f_{X|Y=3}(x)$. Show and explain clearly how you derived your answer. Be sure to define explicitly the relevant range(s) for x and y, and to list all cases.

(d) Are X and Y independent in this example? Give a clear justification for your answer.

Problem 3. (10 points)

Consider a random sample $Y_1, Y_2, \ldots Y_n$ from an Exponential (λ) distribution with

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda} \qquad \text{for } x > 0$$

$$= 0 \qquad \text{elsewhere}$$

Consider the estimator $\hat{\lambda} = n\, Y_{min}$.

(a) Is $\hat{\lambda}$ an unbiased estimator for λ? Clearly justify your answer. Show and clearly explain each step of your solution. [Hint: You might want to find $f_{Y_{min}}(y)$.]

(b) Find the Cramer-Rao lower bound for estimation of λ based on these data. Show and clearly explain each step of your solution.

(c) Compute $Var(\hat{\lambda})$. Show and clearly explain all steps of your solution.

(d) Is $\hat{\lambda}$ an efficient estimator for λ? Clearly justify your answer.