Explain/justify ALL your answers.

1. (*16 points: 4 each part*)

(a) Let $M_X(t) = \mathrm{E}(\exp(tX))$ denote the moment generating function (mgf) of random variable X.
Show that the mgf of kX is $M_X(kt)$ for any constant k.
If X and Y are independent random variables, show that the mgf of $X + Y$ is $M_X(t).M_Y(t)$.

In parts (b), (c) and (d) of this question:
$X_1,\ X_2$ and, X_3 are Normal random variables with mean 0 and variance 3: $X_i \sim N(0, 3)$,
$Y_1,\ Y_2$ and Y_3 are exponential random variables with rate parameter 1/2: $Y_i \sim \mathcal{E}(1/2)$,
W_1, W_2 and W_3 are Gamma random variables with shape parameter 1.5 and rate parameter 1: $W_i \sim G(1.5, 1)$,
and all these nine random variables $(X_i, Y_i, W_i,\ i = 1, 2, 3)$ are independent of each other.

(b) Show that $(1/2)(Y_1 + Y_2 + Y_3)$ has the same distribution as $W_1 + W_2$ and identify this distribution.

(c) Show that $(X_1^2 + X_2^2)$ has the same distribution as $3Y_1$ and identify this distribution.

(d) Show that $(X_1^2 + X_2^2 + X_3^2)$ has the same distribution as $6W_3$ and identify this distribution.

2. (*20 points: 4 each part*)

Suppose that $x_1, ..., x_n$ are the outcomes of n-sample $X_1, ..., X_n$ which are i.i.d from the probability density function $f_X(x; \theta) = \theta x^{\theta-1}/2^\theta$ on $0 \le x \le 2$ (and $f_X(x; \theta) = 0$ otherwise), where $\theta > 0$.

(a) Show that $\mathrm{E}(X_i) = 2\theta/(\theta + 1)$.

(b) Show that the method of moments (MoM) estimator of θ is $\overline{X_n}/(2 - \overline{X_n})$, where $\overline{X_n} = (1/n)\sum_{i=1}^n X_i$.

(c) Show that the likelihood function for θ, based on the n-sample, is $L_n(\theta) = \theta^n(\prod_{i=1}^n x_i)^{\theta-1}/2^{n\theta}$
and identify a sufficient statistic for θ.

(d) Show that the maximum likelihood estimator (MLE) for θ is $1/(\log 2 - (1/n).\sum_{i=1}^n \log(X_i))$.
(You need **not** verify that the 2 nd. derivative of the (log)-likelihood function is negative.)

(e) Which estimator (MoM or MLE) would you prefer to use, and why?