1. X_1, X_2, and X_3 are a sample of three independent realizations from a distribution with mean θ and variance 1.

$$
\begin{array}{llll}
(a) & T_1 = (X_1+X_2+X_3)/3 & & T_2 = (4X_1+3X_2+X_3)/9 \\
& E(T_1) = 3\theta/3 = \theta & & E(T_2) = (4+3+1)\theta/9 = 8\theta/9 \\
& b_1(\theta) = E(T_1)-\theta = 0 & & b_2(\theta) = E(T_2)-\theta = -\theta/9 \\
(b) & \text{var}(T_1) = (1/9).(1+1+1) = 1/3 = 27/81 & & \text{var}(T_2) = (16+9+1)/81 = 26/81 \\
(c) & \text{mse}(T_1) = b_1^2 + \text{var}(T_1) = 0 + 27/81 & & \text{mse}(T_2) = b_2^2 + \text{var}(T_2) = \theta^2/81 + 26/81
\end{array}
$$

(a) From (a) above, T_1 is unbiased, T_2 is not.
(b) From (b) above, T_2 has smaller variance.
(c) From (c) above, they have the same mse if $\theta = \pm 1$.
Otherwise, if $|\theta| < 1$, T_2 has smaller mse; if $|\theta| > 1$, T_1 has smaller mse;

2(a) $E(X^2) = \text{var}(X) + (E(X))^2 = \theta + \theta^2$
(b) $\text{var}(X^2) = E(X^4) - (E(X^2))^2 = \theta(\theta^3 + 6\theta^2 + 7\theta + 1) - \theta^2(\theta+1)^2$
Call this mess $K(\theta)$.
(c) $E(T) = (1/n)\sum E(X_i^2) = E(X^2) = \theta^2 + \theta$ for all θ.
so T is an unbiased estimator of $\theta^2 + \theta$.
(d) From Chebychev's inequality, T_n will be consistent is $\text{mse}(T_n) \to 0$ as $n \to \infty$.
Since T_n is unbiased, $\text{mse}(T_n) = \text{var}(T_n)$ so we need $\text{var}(T_n) \to 0$ as $n \to \infty$.
But $\text{var}(T_n) = (1/n)^2 \sum_{i=1}^n \text{var}(X_i^2)$ since the X_i^2 are independent.
So $\text{var}(T_n) = nK(\theta)/n^2 = K(\theta)/n \to 0$ as $n \to \infty$, and T_n is consistent.

3. (a) The Mom equation is to set $(1/n).\sum_{i=1}^n x_i = E(X_i) = \theta$
That is the MoM estimator of θ is $\overline{X_n} = (1/n)\sum_{i=1}^n X_i$, and the MoM estimator of $\theta^2+\theta$ is $W = \overline{X_n}^2 + \overline{X_n}$.
(b) $E(\overline{X_n}^2) = \text{var}(\overline{X_n}) + (E(\overline{X_n}))^2 = (1/n)\text{var}(X_i) + (E(X_i))^2 = \theta/n + \theta^2$
using the fact that the X_i are independent for the variance term.
(c) $E(W) = E(\overline{X_n}^2) + E(\overline{X_n}) = \theta/n + \theta^2 + \theta$
So $b_W(\theta) = E(W) - (\theta^2 + \theta) = \theta/n \to 0$ as $n \to \infty$.
So W is a biased estimator of $\theta^2 + \theta$, but asymptotically unbiased.