Explain/justify ALL your answers.

There are five questions on this exam.

Each question is 12 points; the 3 or 4 parts of each question have equal weight (4 points or 3 points).

Note: *Questions 1 and 2 go together, and questions 4 and 5 go together, but you do* **not** *need to solve question 1 to use its results in question 2, and do* **not** *need to solve question 4 to use its results in question 5.*

You also do **not** *need to solve earlier parts of questions, to use given results later in the question.*

1. (a) Suppose Y has the Uniform distribution on $-\sqrt{\theta} \le Y \le \sqrt{\theta}$:

that is, the pdf of Y is $f_Y(y;\theta) = 1/(2\sqrt{\theta})$ if $-\sqrt{\theta} \le y \le \sqrt{\theta}$ and $f_Y(y;\theta) = 0$ otherwise.

Show $\mathrm{E}(Y) = 0$, $\mathrm{E}(Y^2) = \theta/3$ and $\mathrm{E}(Y^4) = \theta^2/5$.

(b) Suppose $Y_1, ..., Y_n$ are an n-sample from the Uniform distribution on $-\sqrt{\theta} \le Y \le \sqrt{\theta}$.

Show the Method of Moments (MoM) estimator of θ is $W = (3/n)\sum_{i=1}^n Y_i^2$.

Explain why we use the second moment of the Y_i to form this estimator.

(c) Show that W is an unbiased estimator of θ? What is the variance of the estimator W?

Using Chebychev's inequality, show that W is a consistent estimator of θ.

2, Suppose $Y_1, ..., Y_n$ are an n-sample from the Uniform distribution on $-\sqrt{\theta} \le Y \le \sqrt{\theta}$;

that is, the pdf of each Y_i is $f(y;\theta) = 1/(2\sqrt{\theta})$ if $-\sqrt{\theta} \le y \le \sqrt{\theta}$ and 0 otherwise.

(a) Show that the likelihood for θ may be written as

$$\begin{aligned} L_n(\theta) &= (4\theta)^{-n/2} && \text{if } \theta > \max(Y_i^2) \\ &= 0 && \text{otherwise,} \end{aligned}$$

(b) Identify a sufficient statistic for θ, and show that the maximum likelihood estimator (MLE) for θ is $T = \max(Y_i^2)$.

(c) Given the fact that $\mathrm{E}(T) = n\theta/(n+2)$, find an estimator $T^* = kT$ for some constant k (depending on n) such that T^* is an unbiased estimator of θ.

Which of the three estimators of θ would you prefer to use, the MLE T, this unbiased T^*, or the unbiased MoM estimator W of question 1(b), and why?

(There may not be a unique correct answer/reason here.)

3. Suppose $X_1, ..., X_n$ are an n-sample from the Poisson distribution with mean θ;

$P(X_i = x) = e^{-\theta}\theta^x/x!$, $x = 0, 1, 2, 3....$

(a) Show that the maximum likelihood estimator (MLE) of θ is $T = (1/n)\sum_{i=1}^n X_i$.

(b) If $f(X;\theta) = e^{-\theta}\theta^X/X!$, show that $\mathrm{E}((\frac{d}{d\theta}\log f(X;\theta))^2) = 1/\theta$, and hence determine the Cramér-Rao lower bound on the variance of an unbiased estimator of θ from a sample of size n.

(c) Is T an unbiased estimator of θ? What is the variance of T? Is T an efficient estimator of θ?

4. Suppose $X_1, ..., X_n$ are an n-sample from the exponential distribution with rate parameter θ:
that is, $f_X(x;\theta) = \theta \exp(-\theta x)$ for $0 \leq x < \infty$ (and $f_X(x;\theta) = 0$ for $x < 0$).

(a) Show that $T = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ.

(b) Using the fact that the sum, T, of n i.i.d exponential random varables has a Gamma distribution with shape parameter n, show that $2\theta T$ has a Gamma $G(n, \frac{1}{2})$ distribution.

(c) Using the fact that a $G(n, \frac{1}{2})$ distribution is the same as a chi-squared distribution with $2n$ degrees of freedom (χ^2_{2n}), show how you would use the result of (b) to construct a $(1-\alpha)$-level confidence interval for θ.

(d) Determine the 95% confidence interval if $n = 10$ and T takes the value 12.0. Explain the meaning of the 95% confidence level assigned to this interval.

5. Suppose $X_1, ..., X_n$ are an n-sample from the exponential distribution with rate parameter θ:
$f_X(x;\theta) = \theta \exp(-\theta x)$ for $0 \leq x < \infty$ (and $f_X(x;\theta) = 0$ for $x < 0$). Let $T = \sum_{i=1}^{n} X_i$.

(a) Suppose that a Bayesian statistician gives a prior pdf $\pi(\theta) = 4\theta \exp(-2\theta)$ on $0 \leq \theta < \infty$. Show that, if T takes the value t the Bayesian posterior distribution for θ given $T = t$ is Gamma $G(n+2, t+2)$.

(b) Using the fact that if θ has a $G(n+2, t+2)$ distribution then $2\theta(t+2)$ has a Gamma $G(n+2, \frac{1}{2})$ or $\chi^2_{2(n+2)}$ distribution, show how you would construct a 95% posterior probability interval for θ.

(c) Determine this interval in the case $n = 8$ and $t = 10.0$.

Explain the difference in interpretation between this interval and the one you constructed in question 4.