# STAT 341 Midterm-1: Winter 2010 Feb 1, 2010
# Useful facts

**1. Permutations and combinations**

There are $n! = \prod_{i=1}^{n} i = 1.2.3.4.\ldots n$ permutations of $n$ objects.

There are $\binom{n}{k} = n!/(k!(n-k)!)$ ways of choosing a given $k$ objects from $n$.

**2. Joint and conditional probabilities**

If $C$ and $D$ are any events: $\mathrm{P}(C \bigcup D) = \mathrm{P}(C) + \mathrm{P}(D) - \mathrm{P}(C \bigcap D)$.

The conditional probability of $C$ given $D$ is $\mathrm{P}(C \mid D) = \mathrm{P}(C \bigcap D) / \mathrm{P}(D)$.

$C$ and $D$ are independent if $\mathrm{P}(C \bigcap D) = \mathrm{P}(C).\mathrm{P}(D)$.

**3. Laws and theorems**

Suppose $E_1, \ldots, E_k$ is a partition of $\Omega$. That is $E_i \bigcap E_j$ is empty, and $E_1 \bigcup E_2 \bigcup \ldots \bigcup E_k = \Omega$.

The law of total probability states that: $\mathrm{P}(D) = \sum_{j=1}^{k} \mathrm{P}(D \bigcap E_j) = \sum_{j=1}^{k} \mathrm{P}(D \mid E_j)\ \mathrm{P}(E_j)$

Bayes' Theorem states that: $\mathrm{P}(E_i \mid D) = \mathrm{P}(D \mid E_i)\ \mathrm{P}(E_i)/\mathrm{P}(D)$

**4. Random variables and distributions**

| | discrete (mass) | continuous (density) |
|---|---|---|
| Probability mass/density function | pmf: $\mathrm{P}(X = x) = p_X(x)$ | pdf: $f_X(x)$ |
| Cumulative dist. func. CDF, $\mathrm{P}(X \le x)$ | $F_X(x) = \sum_{w \le x} p_X(w)$ | $F_X(x) = \int_{-\infty}^{x} f_X(w)dw$ |
| Expectation: | $\mathrm{E}(X) = \sum_x x\ \mathrm{P}(X = x)$ | $\int_{-\infty}^{\infty} x f_X(x)dx$ |
| | $\mathrm{E}(g(X)) = \sum_x g(x)\ \mathrm{P}(X = x)$ | $\int_{-\infty}^{\infty} g(x) f_X(x)dx$ |

**provided** the sum/integral converges absolutely.

**For any random variables $X$ and $Y$:**

Variance: $\mathrm{var}(X) = \mathrm{E}((X - E(X))^2) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

Always: $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$, $\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$, $\mathrm{var}(aX + b) = a^2\ \mathrm{var}(X)$.

If $X$ and $Y$ are independent: $\mathrm{var}(X + Y) = \mathrm{var}(X) + \mathrm{var}(Y)$

If $X_1, ..., X_n$ are i.i.d, $\overline{X_n} \equiv (1/n)\sum_{i=1}^{n} X_i$ then $\mathrm{E}(\overline{X_n}) = \mathrm{E}(X_1)$, $\mathrm{var}(\overline{X_n}) = \mathrm{var}(X_1)/n$.

**5. Basics of estimation of parameter $\theta$**

(a) An $n$-sample is a set of $n$ independent data random variables, $Y_1, ..., Y_n$ all from the same distribution (i.i.d), indexed by unknown parameter(s) $\theta$.

(b) A *statistic*, $T$, is any function of $Y_1, ..., Y_n$.

(c) An *estimator* of $g(\theta)$ is a statistic used to estimate $g(\theta)$.

(d) The *estimate* of $g(\theta)$ is the value taken by the estimator in any particular instance.

(e) The Method of Moments (MoM) estimator of one parameter $\theta$ is found by solving
$\overline{Y_n} = (1/n)\sum_{i=1}^{n} Y_i = \mathrm{E}_\theta(Y)$, giving say estimator $W$ for $\theta$. Then estimator of $h(\theta)$ is $h(W)$.

**6. Properties of estimators**

(a) *Bias* of estimator $T$ of $\theta$ is $b_T(\theta) = \mathrm{E}_\theta(T) - \theta$.

(b) *MSE* of estimator $T$ of $\theta$ is $mse_\theta(T) = \mathrm{E}((T - \theta)^2) = \mathrm{var}_\theta(T) + (b_T(\theta))^2$.

(c) An estimator $T_n$ based on an $n$-sample is *asymptotically unbiased* if $b_{T_n}(\theta) \to 0$ as $n \to \infty$.

(d) An estimator $T_n$ based on an $n$-sample is *consistent* if, for all $\epsilon > 0$, $P(|T_n - \theta| > \epsilon) \to 0$ as $n \to \infty$.

(e) Chebychev's inequality: consistency and MSE are related by the fact that, for all $\epsilon > 0$,

$$P(|T_n - \theta| > \epsilon) \le \mathrm{E}((T_n - \theta)^2)/\epsilon^2.$$

**7. Standard distributions:**

| | pmf or pdf | mean | variance |
|---|---|---|---|
| (a) Binomial; $B(n,p)$ | $\mathrm{P}(X=k) = \binom{n}{k} p^k(1-p)^{n-k}$ | $np$ | $np(1-p)$ |
| index $n$, parameter $p$ | $k = 0,1,2,...,n$ | | |
| (b) Geometric; $Geo(p)$; parameter $p$ | $\mathrm{P}(X=k) = p(1-p)^{k-1}$ $k=1,2,3,4,......$ | $1/p$ | $(1-p)/p^2$ |
| (c) Neg. Binomial; $NegB(r,p)$; | $\mathrm{P}(X=k) = \binom{k-1}{r-1} p^r(1-p)^{k-r}$ | $r/p$ | $r(1-p)/p^2$ |
| index $r$, parameter $p$ | $k = r, r+1, r+2, ....$ | | |
| (d) Poisson; $\mathcal{P}o(\mu)$ | $\mathrm{P}(X=k) = \exp(-\mu)\mu^k/k!, \quad k=0,1,2,...$ | $\mu$ | $\mu$ |
| (e) Uniform on $(a,b)$; $U(a,b)$; | $f_X(x) = 1/(b-a), \quad a<x<b$ | $(b+a)/2$ | $(b-a)^2/12$ |
| (f) Normal, $N(\mu,\sigma^2)$ | $f_X(x) = (1/\sqrt{2\pi\sigma^2})\exp(-(x-\mu)^2/2\sigma^2)$ | $\mu$ | $\sigma^2$ |
| (g) Exponential, $\mathcal{E}(\lambda)$ | $f_X(x) = \lambda\exp(-\lambda x)$ | $1/\lambda$ | $1/\lambda^2$ |
| rate parameter $\lambda$ | $0 \le x < \infty$ | | |