Lecture 26; Introduction to the Poisson process Kelly 1.5

26.1 The process

Events occur *randomly and independently* in time, at rate λ.

More formally: the numbers of events N in disjoint time intervals are independent, and the probability distribution of the number of events $N(\ell)$ in an interval depends only on its length, ℓ.

Additionally, $P(N(h) = 1) = \lambda h + o(h)$, $P(N(h) \geq 2) = o(h)$.

26.2 The waiting time T to an event

The waiting time T to an event is $> s$, if there are no events in $(0, s)$.

That is $P(T > s) \; = \; P(N(s) = 0) \; \equiv P_0(s)$.

$$\begin{aligned}
P_0(s+h) &= P_0(s) \times P_0(h) \;=\; P_0(s)(1 - \lambda h - o(h)) \\
P_0(s+h) - P_0(s) &= -\lambda h P_0(s) + o(h) \\
dP_0/P_0 &= -\lambda ds \quad \text{or} \quad \log(P_0) = -\lambda s \;\; \text{with} \;\; P_0(0) = 1 \\
\text{So} \quad P(T > s) &= P_0(s) \;=\; \exp(-\lambda s) \\
\text{So} \qquad F_T(s) &= P(T \leq s) \;=\; 1 - P(T > s) \;=\; 1 - \exp(-\lambda s) \\
\text{So} \qquad f_T(s) &= F_T'(s) \;=\; \lambda \exp(-\lambda s) \quad \text{on} \quad 0 < s < \infty
\end{aligned}$$

That is, regardless of where we start waiting, the waiting time to an event is exponential with rate parameter λ. Recall the "forgetting property" of the exponential: $P(T > t + s | T > t) \; = \; P(T > s)$.

26.3 The number of events $N(s)$ in a time interval length s

Let $N(s)$ be the number of events in interval $(0, s)$ and $P_n(s) = P(N(s) = n)$.

Note from 26.2, $P_0(s) = P(T > s) = \exp(-\lambda s)$. Then

$$\begin{aligned}
P_n(s+h) &= P_n(s)(1 - \lambda h - o(h)) + P_{n-1}(s)(\lambda h + o(h)) \; + \; o(h) \\
P_n(s+h) - P_n(s) &= \lambda h(P_{n-1}(s) - P_n(s)) \; + \; o(h) \\
P_n'(s) &= \lambda(P_{n-1}(s) - P_n(s)) \text{ letting } h \to 0
\end{aligned}$$

Hence from $P_0(s) = \exp(-\lambda s)$ we could determine P_1, P_2, ...

Instead, consider $q_n(s) = \exp(-\lambda s)(\lambda s)^n/n! \; = \; P(\mathcal{P}\text{o}(\lambda s) = n)$.

Then $q_n'(s) \; = \; \exp(-\lambda s)\lambda^n n s^{n-1}/n! \; - \; \lambda \exp(-\lambda s)(\lambda s)^n/n! \; = \; \lambda(q_{n-1}(s) - q_n(s))$.

That is, $P_n(s) \equiv q_n(s)$. That is $N(s)$ is a Poisson random variable with mean λs.

26.4 The conditional distribution of times of events

Suppose we know exactly 1 event occurred in $(0, s)$. At what time T did it occur?

This is a continuous random variable: $P(T = t) = 0$ for every t.

Instead consider the cdf $P(T \leq t)$:

$$\begin{aligned}
F_T(t) &= P(T \leq t \mid N(s) = 1) \;=\; P(T \leq t \;\cap\; N(s) = 1)/P(N(s) = 1) \\
&= P_1(t)P_0(s-t)/P_1(s) \;=\; (\lambda t \exp(-\lambda t)) \exp(-\lambda(s-t))/(\lambda s \exp(-\lambda s)) \;=\; t/s
\end{aligned}$$

So $F_T(t) = t/s$ on $0 < t < s$, or $f_T(t) = 1/s$, $0 < t < s$.

That is T is uniform on the interval $(0, s)$.

Lecture 27: Event counts and waiting times in a Poisson Process

27.1 Sum of independent Poissons is Poisson

Consider a Poisson process rate 1, and time intervals $I_1 = (0, t_1]$, $I_2 = (t_1, t_1 + t_2]$, ...,$I_k = (t_1 + ... + t_{k-1}, t_1 + ...t_k]$. Let X_j be number of events in interval I_j. Then X_j is Poisson with mean t_j, and the X_j are independent. But $\sum_1^k X_j$ is the number of events in interval $(0, t_1 + ... + t_k]$, and so is Poisson with mean $t_1 + ... + t_k$. That is, we have shown that the sum of independent Poisson r.v.s is also Poisson.

27.2 Conditioning on the sum of Poissons

Consider two types of events: call them *red* (R) and *blue* (B). Suppose *red* events occur as a Poisson process rate λ and the blue events occur as a Poisson process rate μ then, if the two processes are independent, the combined events occur as a Poisson process rate $\lambda+\mu$. Total events in time t $(R+B)$ is Poisson mean $(\lambda+\mu)t$. Suppose n total events occur in time t: how many are *red*?

$$\begin{aligned} P(R = k \mid R + B = n) &= P(R = k).P(B = n-k)/P(R + B = n) \\ &= (\exp(-\lambda t)(\lambda t)^k/k!).(\exp(-\mu t)(\mu)^{n-k}/(n-k)!) \ / \ \exp(-(\lambda+\mu)t)((\lambda+\mu)t)^n/n! \\ &= \binom{n}{k}(\frac{\lambda}{\lambda+\mu})^k(\frac{\mu}{\lambda+\mu})^{n-k} \end{aligned}$$

That is, given $R + B = n$, R is binomial $Bin(n, \lambda/(\lambda + \mu))$.

27.3 The minimum of independent exponentials

Using the same idea as in 27.2,
the time to the first R event is exponential $\mathcal{E}(\lambda)$; mean $1/\lambda$.
the time to the first B event is exponential $\mathcal{E}(\mu)$; mean $1/\mu$.
the time to the first either-type event is exponential $\mathcal{E}(\lambda + \mu)$.
That is, the minimum of independent exponential random variables is exponential.

27.4 Time to n th event, T_n, in a Poisson process

Note $T_n \leq t$ if and only if $N(t) \geq n$. So

$$F_{T_n}(t) \ = \ P(T_n \leq t) \ = \ P(N(t) \geq n) \ = \ \sum_{j=n}^{\infty} P(N(t) = j) \ = \ \sum_{j=n}^{\infty} e^{-\lambda t}(\lambda t)^j/j!$$

Diffferentiating, we would obtain the pdf:

$$f_{T_n}(t) \ = \ \frac{d}{dt}F_{T_n}(t) \ = \ \lambda^n e^{-\lambda t} t^{n-1}/(n-1)! \quad \text{on } 0 \leq t < \infty$$

This pdf is called the Gamma $G(n, \lambda)$ distribution (next quarter!!)
That is T_n has the $G(n, \lambda)$ distribution.

27.5 The sum of independent exponential random variables

Let X_i be time from $(i-1)^{th}$ event to i^{th} event: $T_n \ = \ X_1 + X_2 + X_3 + ... + X_n$.
But $T_n \ \sim G(n, \lambda)$, and X_i are independent $\mathcal{E}(\lambda)$.
Hence we have shown that the sum of independent exponential rvs. is a Gamma r.v.
Note: λ^{-1} is scale parameter in both distributions.
Note $\mathrm{E}(X_i) = 1/\lambda$, $\mathrm{E}(T_n) = n/\lambda$: recall expecations *always* add.
Note $\mathrm{var}(X_i) = 1/\lambda^2$, $\mathrm{var}(T_n) = n/\lambda^2$: recall variances add *for independent r.vs.*

Lecture 28: Poisson process examples

Seattle is in its worst snow storm in 30 years, but the Metro buses are keeping going. There is no schedule, but the #48 arrives at my stop as a Poisson process rate 4 per hour, the #72 arrives as a Poisson process rate 3 per hour and the #373 arrives at rate 2 per hour. (I can get to campus on any of these buses, but the #48 is most convenient, as I need to get down to Genome Sciences.)

28.1 Waiting times

(a) What is the pdf of the waiting time to the next #48 bus? What is expected waiting time?

(b) What is the pdf of the waiting time to the next (any #) bus? What is the expected waiting time? What is the standard deviation?

(c) I arrive and see I have just missed a bus: What is the pdf of the waiting time to the next (any #) bus? What is the expected waiting time?

(d) I arrive at the bus stop: how long is it since the last bus? (the pdf and the expectation).

(e) Suppose the third bus to arrive will be the #48. What is the expectation of the waiting time to this bus? What is the variance?

28.2 Number of buses in given time intervals

(a) What is the pmf of the number of #48 that will arrive in the next half hour? What is the expected number? What is the variance?

(b) What is the pmf of the total number of buses to arrive in the next two hours? What is the standard deviation?

(c) What is the probability that 4 #48 buses will come in the next 15 minutes, but then none in the 45 minutes after that?

(d) What is the probability that in the next hour, there will be exactly 9 buses arrive.

(e) What is the probability that in the next hour, there will be exactly 4 #48, 3 #72 and 2 #373 buses arrive

28.3 Conditional probabilities

(a) What is the probability that the next bus to arrive will be a #48?

(b) Given that the last 6 buses have **not** been a #48, what is the probability that the next bus to arrive will be a #48?

(c) Given that exactly 4 #48 buses will come in the next hour, what is the probability that all 4 will come in the next 15 minutes?

(d) Given that exactly 1 bus will come in the next 10 minutes, what is the pdf of my waiting time? What is the mean? What is the standard deviation?