Lecture 11: Discrete random variables Kelly 1.2.8, 3.1 (There are no notes for lecture 10: Midterm-1)

11.1 Definitions and examples

(i) **Definition:** A random variable X is a real-valued function on the sample space.

(ii) Example: the number of heads in 10 tosses of a fair coin.

(iii) Example: the number of children of blood-type A out of n, from marriages of $AB \times O$ parents.

(iv) Example: the number of red pea plants out of n, from crosses of $RW \times RW$ pea plants.

(v) Example: the number of red fish, in sampling k fish without replacement, from a pond in which there are N fish of which n are red.

(vi) Example: the number of traffic accidents in a large city in a year.

11.2: Discrete Random variables

(i) **Definition:** A random variable X is *discrete* if it can take only a discrete set of values.

(ii) Example (ii) above: X takes a value in $\mathcal{X} = \{0,1,2,3,4,5,6,7,8,9,10\}$.

(iii) Example (iii) above: X takes a value in $\mathcal{X} = \{0,1,2,...,n\}$.

(iv) Example (iv) above: ditto.

(v) Example (v) above: X takes values in $\mathcal{X} = \{\max(0, k+n-N), ...\min(k,n)\}$.

(vi) Example (vi) above: X takes values in $\mathcal{X} = \{0,1,2,..\} = \{0\} \cup \mathcal{Z}_+$.

11.3 Probability mass function

(i) **Definition:** The *probability mass function* (p.m.f.) of a discrete random variable X is the set of probabilities $P(X = x)$ for each of the values $x \in \mathcal{X}$ that X can take.

(ii) Example (ii): $P(X = x) = \binom{10}{x}(1/2)^{10}$ for $x = 0, 1, 2, ...10$.

(iii) Example (iii): $P(X = x) = \binom{n}{x}(1/2)^{n}$ for $x = 0, 1, 2, ...n$.

(iv) Example (iv): $P(X = x) = \binom{n}{x}(3/4)^{x}(1/4)^{n-x}$ for $x = 0, 1, 2, ...n$.

(v) Example (v): $P(X = x) = \binom{n}{x}\binom{N-n}{k-x}/\binom{N}{k}$ for $x = \max(0, k+n-N), ...\min(k,n)$.

(vi) Example (vi): $P(X = x) = \exp(-\lambda)\lambda^x/x!$ for $x = 0, 1, 2, 3, 4.....$

11.4 Properties of probability mass functions

(i) $P(X = x) \geq 0$ for each $x \in \mathcal{X}$. (For discrete random variables, in fact $P(X = x) > 0$ for each $x \in \mathcal{X}$.)

(ii) $\sum_x P(X = x) = 1$ where the sum is over all $x \in \mathcal{X}$.

(iii) Events $\{X = x\}$ for each x are disjoint: $P(X \in B) = \sum_{x \in B} P(X = x)$.

11.5 Names of some standard probability mass functions

(i) *Binomial:* Examples (ii), (iii), and (iv) are Binomial random variables, $Bin(n,p)$, with index n and *parameter* $p = (1/2)$, $(1/2)$ and $(3/4)$ respectively.

(ii) *Bernoulli;* If $n = 1$, examples (iii) and (iv) are Bernoulli random variables, $Bernoulli(p)$.

(iii) *Multinomial:* If there are more that two types (for example, number of each of types A, B, AB and O in sample size n from a population) then we have a Multinomial random variable.

(iv) *Hypergeometric:* In example (v), X is a Hypergeometric random variable.

(v) *Poisson:* In example (vi), X is a Poisson random variable, with parameter λ, $\mathcal{P}o(\lambda)$.

Lecture 12: Continuous random variables Kelly 1.3, 3.1

12.1 Definitions and examples

(i) **Definition:** A *continuous* random variable X is one that takes values in $(-\infty, \infty)$. That is, in principle: some values may be impossible.

(ii) Example: A random number between a and b: values in the interval (a, b).

(iii) Example: The waiting time until the bus arrives: values in $(0, \infty)$.

(iv) Example: The height of an individual, relative to the population average: values in $(-\infty, +\infty)$ (in principle).

12.2 The probability density function: definition and basic properties.

(i) **Definition:** The *probability density function* (p.d.f.) of a continuous random variable X is a non-negative function f defined for all values x in $(-\infty, \infty)$ such that for any subset B for which $X \in B$ is an event

$$P(X \in B) \;=\; \int_B f(x)\;dx$$

(ii) In fact, events can be made up of unions and intersections of intervals of the form $(a, b]$:

$$P(X \in (a, b]) \;=\; P(a < X \le b) \;=\; \int_a^b f(x)\;dx$$

(iii) Note the value at the boundary does not matter:

$$P(X = a) \;=\; \int_a^a f(x)\;dx \;=\; 0 \quad \text{for any continuous random variable.}$$

(iv) Note: $f(x) = 0$ is possible for some x-values (see the p.m.f).
For example, if $X \ge 0$ (as in the waiting-time example), $f(x) = 0$, if $x < 0$.

(v) X takes some value in $(-\infty, \infty)$ so

$$1 \;=\; P(-\infty < X < \infty) \;=\; \int_{-\infty}^{\infty} f(x)\;dx$$

(vi) The *cumulative distribution function* of X is

$$F(x) \;=\; P(-\infty < X \le x) \;=\; \int_{-\infty}^{x} f(z)\;dz$$

12.3 Examples: of the probability density function

(i) In general: any non-negative real-valued function f on the real line such that $\int_{-\infty}^{\infty} f(x)\;dx \;=\; 1$.

(ii) Example (ii) above: *Uniform p.d.f*: $U(a, b)$.

$$f(x) \;=\; \frac{1}{(b-a)} \text{ for } a \le x \le b \text{ and } f(x) \;=\; 0 \text{ otherwise.}$$

(iii) Example (iii) above: *Exponential p.d.f* (with rate parameter λ): $\mathcal{E}(\lambda)$.

$$f(x) \;=\; \lambda \exp(-\lambda x) \text{ for } x \ge 0 \text{ and } f(x) \;=\; 0 \text{ if } x < 0.$$

(iv) Example (iv) above: *Normal p.d.f* (with parameters μ and σ^2): $N(\mu, \sigma^2)$.

$$f(x) \;=\; \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \text{ for } -\infty < x < \infty.$$