Lecture 4: More counting examples; Kelly 1.2, A.4

4.1 Binomial counts and Stirling's formula

(i) Suppose there are N equiprobable outcomes in Ω.

Suppose event E is true for R of these outcomes. Then $P(E) = R/N$.

(ii) An AB parent and an O parent can have an A child or a B child.

Suppose they have n children: there are 2^n possible sequences of A and B children.

Assume these are equiprobable. (In fact, they are.)

$\binom{n}{k}$ of these sequences have k A children. $\quad P(k\ A \text{ children out of } n) = \binom{n}{k}/2^n$.

(iii) $n!$ can be approximated for large n by $\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}$. Also $\binom{n}{k}$ is largest when $k \approx n/2$.

Then, for large n,

$$\binom{n}{n/2} = \frac{n!}{(n/2)!(n/2)!} \approx \frac{\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}}{\sqrt{2\pi}(n/2)^{(n/2)+\frac{1}{2}}e^{-(n/2)} \times \sqrt{2\pi}(n/2)^{(n/2)+\frac{1}{2}}e^{-(n/2)}}$$

$$= (1/\sqrt{2\pi})2^{n+1}n^{n+\frac{1}{2}-(n/2)-\frac{1}{2}-(n/2)-\frac{1}{2}} = (1/\sqrt{2\pi})(2/\sqrt{n})2^n$$

$$\text{Or } P((n/2)\ A \text{ children out of } n) = \binom{n}{n/2}(\frac{1}{2})^n = 1/\sqrt{2\pi(n/4)}$$

This result will have meaning later when we discuss approximating Binomial probabilities by the Normal probability distribution.

4.2 The binomial theorem; Kelly A.4.7

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$$

Note in each bracket we choose x or y. There are 2^n sequences.

The number of sequences in which there are k choices of x is $\binom{n}{k}$, and each has value $x^k y^{n-k}$.

The case of 4.1 (ii) is a special case when $x = y = \frac{1}{2}$.

An alternative proof is by induction using result of 2.2 (iv).

4.3 Sampling with and without replacement (Kelly 1.2, P. 23)

(i) Draw 3 cards from 52-card pack. $E = \{\text{draw at least one face card}\}$. Note E^c (no face card) is easier.

With replacement: $P(\text{no face card}) = (40 \times 40 \times 40)/(52 \times 52 \times 52) = 0.455$.

$\qquad P(\text{at least 1 face card}) = 1 - 0.455 = 0.545$.

Without replacement: $P(\text{no face card}) = (40 \times 39 \times 38)/(52 \times 51) \times 50) = 0.447$.

(ii) **The birthday problem:** ignore Feb 29, and assume other days equiprobable.

In k people, E is event that at least 2 share a birthday. E^c is the event that all bithdays are different.

$P(E^c) = 365.364.363.362......(365-k+1)\ /\ 365.365.365.....365 = 365!\ /\ (365-k)! \times (1/365)^k$

$k = 2$, $P(E) = 1/365$; $k = 23$, $P(E) \approx 0.5$; $k = 45$, $P(E) = 0.94$.

(iii) **Hypergeometric probabilities:** N fish in a pond; n are red, $N - n$ are blue.

Sample k fish without replacement. What is the probability x are red?

Total number of outcomes is $\binom{N}{k}$. Number of ways of choosing the x red fish is $\binom{n}{x}$, and the $(k - x$ blue fish is $\binom{N-n}{k-x}$. Probability is $\binom{n}{x} \times \binom{N-n}{k-x} / \binom{N}{k}$

Lecture 5: Conditional probability and independence Kelly 2.2, 2.4

5.1 Conditional probability (Kelly 2.2)

(i) Idea: revise probabilities in light of restrictions or partial information.

(ii) Definition: probability of D given E is $P(D \mid E) = P(D \cap E)/P(E)$, provided $P(E) > 0$.

(iii) Example: Our AB and O parents have three children.

$\Omega = \{AAA, AAB, ABA, ABB, BAA, BAB, BBA, BBB\}$ and these are equiprobable (1/8 each).

Let D be at least 2 A children: $P(D) = 4/8 = 1/2$. Note also $P(\text{first child is } A) = 4/8 = 1/2$.

$P(D \mid \text{first child is } A) = P(D \cap \text{ first child is } A)/P(\text{first child is } A) = (3/8)/(4/8) = 3/4$.

5.2 The chain rule (Kelly 2.2.10, 2.2.11) If $P(E_1 \cap E_2 \cap ... \cap E_n) > 0$,

$$P(E_1 \cap E_2 \cap ... \cap E_n) = P(E_1).P(E_2|E_1).P(E_3 \mid E_1 \cap E_2)....P(E_n \mid E_1 \cap E_2 \cap ...... \cap E_n)$$

To prove, just write it out:

$$P(E_1).P(E_2|E_1).P(E_3 \mid E_1 \cap E_2)....P(E_n \mid E_1 \cap E_2 \cap ...... \cap E_n) =$$
$$P(E_1)\frac{P(E_2 \cap E_1)}{P(E_1)}\frac{P(E_3 \cap E_2 \cap E_1)}{P(E_2 \cap E_1)}\cdots \qquad \cdots\frac{P(E_n \cap ... \cap E_3 \cap E_2 \cap E_1)}{P(E_{n-1} \cap .. \cap E_2 \cap E_1)}$$

This looks messy, but in fact we have already used it in examples of sampling without replacement.

Example: No face card in 3 cards, when sampling without replacement. (40/52).(39/51).(38/50).

5.3 Bayes' formula: Kelly 2.3

Assume $P(D)$ and $P(E)$ are both > 0. Then, by definition,

$$P(D \mid E)\ P(E) = P(D \cap E) = P(E \mid D)\ P(D) \quad \text{or} \quad P(D \mid E) = P(E \mid D)\ P(D)\ /\ P(E)$$

Note also, from the law of total probability, we could compute $P(E)$ as

$$P(E) = P(E \cap D) + P(E \cap D^c) = P(E|D)P(D) + P(E \mid D^c)(1 - P(D))$$

Example: $P(\text{first child is } A \mid D \equiv \text{“at least } 2A \text{ children”})$

$= P(D \mid \text{first is } A)P(\text{first is } A)/P(D) = (3/4) \times (1/2)\ /\ (1/2) = 3/4$.

For better examples see Lecture 6: $P(D|A) = P(A|D)$ **ONLY BECAUSE** $P(D) = P(A)$.

5.4 Independent events (Kelly 2.4)

(i) Definition: E and F are independent if $P(E \cap F) = P(E) \times P(F)$.

(ii) Interpretation: Knowing F happens does not affect the probability of E: $P(E|F) = P(E)$.

(iii) If E and F are independent, $P(E \cup F) = P(E) + P(F) - P(E).P(F)$ so

$P(E^c \cap F^c) = 1 - P(E \cup F) = 1 - P(E) - P(F) + P(E).P(F) = (1 - P(E)).(1 - P(F)) = P(E^c).P(F^c)$.

That is E^c and F^c are independent. (So are E^c and F, and E and F^c).

Example: $P(A) = 0.36$, $P(B) = 0.2$, $P(AB) = 0.08, P(O) = 0.36$. $P(\text{have antigen } A) = P(A) + P(AB) = 0.44$, $P(\text{have antigen } B) = P(B) + P(AB) = 0.28$.

$0.44 \times 0.28 = 0.1232 \neq 0.08 = P(AB)$: Having antigen A is NOT independent of having antigen B.

$P(\text{have antigen } A \mid \text{have antigen } B) = P(AB)/P(\text{have antigen } B) = 0.08/0.28 = 0.286$

$< 0.44 = P(\text{have antigen } A)$

Knowing a person has the B antigen decreases the probability they have the A antigen.

Lecture 6: Probability Examples

6.1 More conditional probabilities

(i) Mendel discovered that every pea plant has two *factors* or *alleles* for flower color: R (red) and W (white). Each plant has RR or RW and is red, or WW and is white.

We cross two plants which we know are RW. The offspring plants are *independent* and each is RR, RW or WW with probabilities 1/4, 1/2, 1/4.

What is the probability an offspring is RR, given it is red.

Answer: $P(\text{red}) = 3/4$, $P(RR \cap \text{red}) = P(RR) = 1/4$, so $P(RR \mid \text{red}) = (1/4)/(3/4) = 1/3$.

(ii) **Testing for a rare disease** (Kelly: Exx 2.3.7)

Suppose we have a quite effective test, so $P(+ \mid \text{disease}) = 0.99$, and test is quite accurate, so $P(+ \mid \text{no disease}) = 0.02$). Now suppose the frequency of the disease is 0.001.

$$\begin{aligned} P(+\text{ test result}) &= P(+ \mid \text{disease})P(\text{disease}) + P(+ \mid \text{no disease})P(\text{no disease}) \\ &= 0.99 \times 0.001 + 0.02 \times (1 - 0.001) = 0.00099 + 0.0198 = 0.02097 \\ P(\text{disease} \mid +) &= P(+ \mid \text{disease})P(\text{disease})/P(+) = 0.99 \times 0.001/0.02097 = 0.047 \end{aligned}$$

Less than 5% of people testing positive actually have the disease!

The probability of spots given measles is large: the probability of measles given spots is small.

The probability of 4 legs given elephant close to 1: the probability of elephant given 4 legs is close to 0.

6.2 Updating with more information

(i) Allele frequencies in a population of pea plants.

Suppose the R allele has frequency 0.3, and W 0.7, and the types of the two alleles in an individual are independent. What is the probability a red-flowered pea-plant is RR?

Solution: By independence, $P(RR) = 0.3 \times 0.3 = 0.09$, $P(WW) = 0.7 \times 0.7 = 0.49$,

$$P(RW) = 1 - 0.09 - 0.49 = 0.42.$$

Overall $P(\text{red}) = P(RR) + P(RW) = 0.09 + 0.42 = 0.51$,

so then $P(RR \mid \text{red}) = P(RR \cap \text{red})/P(\text{red}) = 0.09/0.51 = 0.176$.

(ii) The red pea plant is crossed to a white one (WW). The first offspring has red flowers. What now is the probability the red parent plant is RR?

Solution: $P(\text{red offspring} \mid RR \text{ parent}) = 1$, $P(\text{red offspring} \mid RW \text{ parent}) = (1/2)$.

$$\begin{aligned} P(\text{red offspring}) &= P(\text{red offspring} \mid RR)P(RR) + P(\text{red offspring} \mid RW)P(RW) \\ &= 1 \times 0.176 + (1/2)(1 - 0.176) = 0.588. \end{aligned}$$

$$\begin{aligned} P(\text{red parent is } RR \mid \text{red offspring}) &= P(\text{red offspring} \mid \text{red parent is } RR) \times P(\text{red parent is } RR)/P(\text{red offspring}) \\ &= 1 \times 0.176/0.588 = 0.299 \approx 0.3. \end{aligned}$$

6.3 Independent events

(i) Back to the two RW parent pea plants of 6.1 (i):

Each offspring is red with probability 3/4, and white with probability 1/4, and the colors of offspring pea plants are *independent*. We grow up 10 offspring.

What is the probability all 10 are red: answer $(3/4)^{10} = 0.0563$

What is the probability of at least one red: answer $1 - P(\text{all white}) = 1 - (1/4)^{10} = 0.9999$.