Lecture 1: Sample spaces and events: Kelly 1.1, 1.2

1.1 Sample spaces

The *sample space* Ω is the set of all possible outcomes of an experiment.

One and only one outcome can occur.

1.2 Examples

(i) Child is boy or girl: $\Omega = \{\text{boy, girl}\}$

(ii) Toss of one die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

(iii) Number of traffic accidents: $\Omega = \{0, 1, 2, 3, 4, ...\} = \{0\} \cup \mathcal{Z}_+$.

(iv) Time waiting for the bus: $\Omega = (0, \infty) = \Re_+$, the positive half line.

1.3 Events

Any subset E of Ω is a *event*.

(The book says this: it is OK for countable sample spaces, but an oversimplification for a space like $\Re_+$.)

1.4 Combining events

(i) If E is an event, not-E (the *complement* of E: written E^c) is an event.

(ii) If E is an event and F is an event, then "E and/or F" is an event. "E and/or F" is written $E \cup F$.

(iii) If E_1, E_2, ... are events then $E_1 \cup E_2 \cup E_3$.... is an event. (Countable unions.)

(iv) If E is an event and F is an event, then "E and F" is an event. "E and F" is written $E \cap F$.

(v) If E_1, E_2, ... are events then $E_1 \cap E_2 \cap E_3$.... is an event. (Countable intersections.)

1.5 More events:

(i) The empty set Φ is an event: so $\Omega = \Phi^c$ is an event.

If $E \cap F = \Phi$, E and F are *disjoint* also known as *mutually exclusive*.

(ii) E and E^c are *disjoint events*: $E \cup E^c = \Omega$, $E \cap E^c = \Phi$.

(iii) Events E_1, E_2 .....E_k are *mutually exclusive* if $E_i \cap E_j = \Phi$ for all pairs (i, j) $(i, j = 1, ..., k,\ i \neq j)$.

(iv) Events E_1, E_2 .....E_k are *exhaustive* if $E_1 \cup E_2 \cup .... \cup E_k = \Omega$.

(v) If Ω is discrete, the elements of Ω are a set of mutually exclusive and exhaustive events.

1.6 A genetic example: The ABO blood types.

We can be blood type A, B, AB or O.

Let our "experiment" be finding the blood types of two children in a family.

Then $\Omega = \{(i, j); i, j = A, B, AB, O\}$.

Let E_1 be the first child is type A. Let E_2 be the first child is type O.

Let E_3 be the second child is type B.

Let E_4 be at least one child is type A. Let E_5 be at most one child is type A.

Let E_6 be the two children have the same blood type.

Let E_7 be the two children have different blood types.

Which pairs of events are *complements*?

Which pairs of events are *disjoint*?

Which pair of events is *mutually exclusive and exhaustive*?

What is the intersection of E_4 and E_5?

Lecture 2: Permutations and combinations: Kelly 1.1, 1.2, 2.1

2.1 Basic principle of counting

If an experiment has k steps, and if earlier choices do NOT limit later ones, then if step-1 can be done in n_1 ways, step-2 in n_2 ways, ... step-k in n_k ways,
then there are $n_1 \times n_2 \times ... \times n_k$ possible outcomes for (step-1, ..., step-k).

Corollary: There are 2^k subsets of a set size k.

Proof: Each element i, $i = 1, ..., k$ can be chosen, or not: $n_i = 2$, $i = 1, ..., k$.
So total possible is $2 \times 2 \times .... \times 2 = 2^k$.
Note: for proper (not Ω), non-empty (not Φ) subsets, there are $2^k - 2$.

2.2 Permutations and combinations

(i) The number of ways of ordering n distinct objects is $n(n-1)(n-2)....3.2.1 \;=\; n!$ (n-factorial).

(ii) The number of ways of choosing k distinct objects, in order, from n is $n(n-1)....(n-k+1) \;=\; n!/(n-k)!$.

(iii) If we do not care about the order in which the k objects are selected, there are $k!$ selections that give the same *combination*.

That is there are $n!/((n-k)!k!)$ distinct *combinations*: this is often written ${}_nC_k$ or $\binom{n}{k}$.

(iv) A useful formula:

$$\binom{n}{k} \;=\; \binom{n-1}{k-1} + \binom{n-1}{k}$$

Consider the number of choices that do and do not contain the particular object "1".

2.3 Multinomial combinations

Number of ways of arranging n_1 objects type-1, n_2 objects type-2, ... n_k objects type-k,
where $n_1 + n_2 + .. + n_k \;=\; n$:

Choose the n_1 positions for type 1: $\binom{n}{n_1} \;=\; n!/(n_1!(n-n_1)!)$.

Now out of the remaining $(n - n_1)$ positions choose n_2 for type-2:

number of ways $= \binom{n-n_1}{n_2} \;=\; (n-n_1)!/(n_2!(n-n_1-n_2)!)$. etc. ...

Total number of ways is

$$\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\frac{(n-n1-n_2)!}{n_3!(n-n_1-n_2-n_3)!}\cdots\frac{(n-n1-n_2-...-n_{k-1})!}{n_k!0!} \;=\; \frac{n!}{n_1!\;n_2!....n_k!}$$

Example: Twelve students go to donate blood: 5 are type A, 2 are type B, one is AB, and 4 are type O. How many different orderings of the types of blood in the 12 blood donation tubes are there?

Answer: $12!/(5! \times 2! \times 1! \times 4!) \;=\; (12.11.10.9.8.7.6)/(2.4.3.2) \;=\; 12.11.10.9.7 \;=\; 914,760$.

Lecture 3: Probabilities of Events: Kelly 1.2, 2.1

3.1 Probability axioms

For each event E we assume we can assign a number $P(E)$ which satisfies the following three axioms:

(i) $P(E) \geq 0$ for every event E.

(ii) $P(\Omega) = 1$

(iii) If $E_1, E_2, ...$ are *mutually exclusive* $P(E_1 \cup E_2 \cup E_3 \cup ....) \ = \ P(E_1) + P(E_2) + P(E_3) + ...$

Note: for a countable sample space, each outcome (element of Ω) has a probability, and each even is a union of outcomes, with probability the sum of the probabilities of the outcomes.

3.2 Probability interpretation as a limiting frequency

A useful *interpretation* of $P(E)$ is that it is the proportion of times an outcome in E occurs in a large number of repetitions of the same experiment with outcomes in the sample space Ω.

Example: Sampling an individual from a very large population.

$\Omega = \{A, B, AB, O\}$.

$P(A)$ can be interpreted as the proportion of A blood-type individuals in the population. If we repeat the sampling of an individual again, and again, the proportion of times we observe the individual to have blood type A is $P(A)$.

For the USA population, roughly, $P(A) = 0.36$, $P(B) = 0.20$, $P(AB) = 0.08$, and $P(O) = 0.36$.

$P(\text{antigen A on red blood cells}) \ = \ P(\{A\} \cup \{AB\}) \ = \ P(A) + P(AB) \ = \ 0.44$ for this example.

3.3 Basic probability formulae

(i) $\Omega \ = \ E \cup E^c$, $E \cap E^c \ = \ \Phi$, so $P(E^c) + P(E) \ = \ P(\Omega) \ = \ 1$, or $P(E^c) \ = \ 1 - P(E)$.

This also shows $P(E) \leq 1$, since all probabilities are non-negative.

(ii) $E \cup F \ = \ E \cup (E^c \cap F)$, so $P(E \cup F) \ = \ P(E) + P(E^c \cap F)$.

So $P(E \cup F) + P(E \cap F) \ = \ P(E) + P(E^c \cap F) + P(E \cap F) \ = \ P(E) + P(F)$,

or $P(E \cup F) \ = \ P(E) + P(F) - P(E \cap F)$.

3.4 Two important probability formulae

(i) Law of total probability

Suppose $E_1, E_2, ...,$ form a *partition* of Ω.

That is, $E_1, E_2, ...$ are mutually exclusive and exhaustive.

That is, $E_i \cap E_j \ = \ \Phi$ (disjoint), and $\Omega = E_1 \cup E_2 \cup ....$

Then for any event F, $F \ = \ \cup_i (F \cap E_i)$, $P(F) \ = \ \sum_i P(F \cap E_i)$.

Special case: if E_i is ith outcome in a countable Ω, $F \cap E_i \ = \ E_i$ or $F \cap E_i \ = \ \Phi$, and $P(F) = \sum_{i \in F} P(E_i)$.

(ii) The inclusion and exclusion formula

$$
\begin{aligned}
P(D \cup E) &= P(D) + P(E) - P(D \cap E). \\
P(C \cup D \cup E) &= P(C) + P(D) + P(E) - P(C \cap D) - P(D \cap E) - P(C \cap E) + P(C \cap D \cap E). \\
P(E_1 \cup E_2 \cup ... \cup E_k) &= P(E_1) + P(E_2) + ...P(E_k) \\
&\quad -P(E_1 \cap E_2) - \text{all the other 2-way} \\
&\quad +P(E_1 \cap E_2 \cap E_3) + \text{all the other 3-way} \\
&\quad -P(E_1 \cap E_2 \cap E_3 \cap E_4) - \text{all the other 4-way} \\
&\quad ...... \pm P(E_1 \cap E_2 \cap .... \cap E_k).
\end{aligned}
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