Homework 9; Due 11.30 a.m. Wed Dec 9.

1.4.4 (a) Rate 4/day, gives expected value 1/6 in any hour. Probability Poisson with mean 1/6 is more than 2 is $1 - \exp(-1/6)(1 + (1/6) + (1/6)^2/2) \; = \; 0.00068$.

(b) Rate 4/day, gives expectation 4 in any day. Probability exactly 4 is $4^4 \exp(-4)/4! \; = \; 0.195$

(c) Rate 4/day gives expectation 28 in 1 week. Probability exactly 28 is $28^{28} \exp(-28)/28! \; = \; 0.0752$.

(d) None tomorrow (expectation 4) is $\exp(-4)$, then 3 in 12 hours (expectation 2) is $\exp(-2)2^3/3!$, and these are independent events, so overall $\exp(-6)8/6 \; = \; 0.0033$.

1.4.8 None in three years, means waiting time T to first is more than 3 years, and time to event is exponential parameter (1/mean) 3.1418, so probability is $P(T > 3) \; = \; \exp(-3 \times 3.1418) \; = \; 0.00008$.

Easier, number in 3 years is a Poisson random variable X with mean $3 \times 3.1418 \; = \; 9.425$.

$P(X = 0) = \exp(-9, 425) \; = \; 0.00008$.

2.2.16 $P(3 \text{ events in 2 minutes}) \; = \; e^{-2\lambda}(2\lambda)^3/3!$

$k = 0, 3$: Probability 0 in 1 minute, and 3 in other is $e^{-\lambda} \times e^{-\lambda}\lambda^3/3!$,

so conditional probability is $e^{-\lambda} \times e^{-\lambda}\lambda^3/3!/e^{-2\lambda}(2\lambda)^3/3! \; = \; 1/2^3 \; = \; 1/8$.

$k = 1, 2$: Probability 1 in 1 minute, and 2 in other is $\lambda e^{-\lambda} \times e^{-\lambda}\lambda^2/2!$,

so conditional probability is $\lambda e^{-\lambda} \times e^{-\lambda}\lambda^2/2!/e^{-2\lambda}(2\lambda)^3/3! \; = \; 3/2^3 \; = \; 3/8$.

2.4.11 (a) Let X be the location of one point. $P(X \le a) \; = \; F_X(a) \; = \; a$ if $X \sim U(0, 1)$. The points are independent, so the probability all are in $(0, a)$ is a^n.

(b) For the Poisson process, the probability of n points in $(0, 1)$ is $\exp(-\lambda)\lambda^n/n!$.

The probability of n points in $(0, a)$ is $\exp(-a\lambda)(a\lambda)^n/n!$.

The probability of 0 points in $(a, 1)$ is $\exp(-(1 - a)\lambda)$.

So the conditional probability is $\exp(-a\lambda)(a\lambda)^n/n! \times \exp(-(1 - a)\lambda) \; / \; \exp(-\lambda)\lambda^n/n! \; = \; a^n$.

3.3.9 X is number in first hour, Y is number in 2 hours, rate $\lambda \; = \; 1.6$/hr.

(a) For Y, expectation is 3.2, $P(Y \ge 2) \; = \; 1 - P(Y = 0) - P(Y - 1) \; = \; 1 - \exp(-3.2)(1 + 3.2) \; = \; 0.829$.

(b) For $P(Y \ge 2|X = 0)$, we need ≥ 2 in second hour (expectation 1.6).

Probability is $1 - \exp(-1.6)(1 + 1.6) \; = \; 0.475$.

(c) For $P(Y = X)$ we need 0 in the second hour (expectation 1.6). Probability is $\exp(-1.6) = 0.202$.

(d) $Y - X$ is the number in the second hour, so is Poisson with expectation 1.6:

$P(Y - X \; = \; k) \; = \; \exp(-1.6)(1.6)^k/k!$ for $k = 0, 1, 2, 3, ...$

8.3.6 (b,c,e) Customers come as Poisson process, rate 8/minute. X is arrival time of 100 th. customer. We want to know $P(X \le 10)$.

(b) Let Y be number of customers in 10 minutes; Y is Poisson with mean 80. Note $X \le 10$ if and only if $Y \ge 100$. $P(Y \ge 100) \; = \; 1 - P(Y < 100) \; = \; 1 - e^{-80} \sum_{k=0}^{99} 80^k/k!$

(c) X is sum of 100 exponentials, each mean 1/8, variance 1/64. So $\mathrm{E}(X) = 100/8 = 12.5$, $\mathrm{var}(X) = 100/64$, so $Z = (X - 12.5)/\sqrt{100/64} \; = \; 0.8(X - 12.5)$ is approx $N(0, 1)$.

$P(X \le 10) \; = \; P(Z \le -0.8 \times 2.5 = -2) \; \approx \; 0.0228$.

(e) $\mathrm{E}(Y) = 80$, and $\mathrm{var}(Y) = 80$ since Y is Poisson. $Z = (Y - 80)/\sqrt{(}80)$ is approx $N(0, 1)$.

$P(Y \ge 100) \; = \; P(Z \ge 19.5/\sqrt{80} = 2.23) \; = \; 0.0146$. (using continuity correction).