Homework 6; Due 11.30 a.m. Fri Nov 13.

4.1.10, $\mathcal{X} = \{1, 2, 1000\}$;
$\mathrm{E}(X) = 1 \times P(X = 1) + 2 \times P(X = 2) + 1000 \times P(X = 1000) = 0.25 + 1.0 + 250 = 251.25$
(This is a trivial question, but makes an important point.)

4.1.11, (a) $f_X(x) = \lambda \exp(-\lambda x)$ on $x > 0$. $\mathrm{E}(X) = \int_0^\infty \lambda x \exp(-\lambda x) dx = 1/\lambda$
$P(X > 1/\lambda) = \int_{1/\lambda}^\infty f_X(x) dx = [-\exp(-\lambda x)]_{1/\lambda}^\infty = \exp(-1) = 0.3679$.
(b) $P(X > b) = [-\exp(-\lambda x)]_b^\infty = \exp(-\lambda b)$.
So we need $\exp(-\lambda m) = 1/2$ or $\exp(\lambda m) = 2$ or $m = \log_e(2)/\lambda = 0.693/\lambda < 1/\lambda = \mathrm{E}(X)$.

4.2.1, $f_X(x) = 3x^2/8$ on $0 < x < 2$; note this does integrate to 1!
(a) $\mathrm{E}(X) = \int_0^2 3x^3/8 dx = (3/8) \times [x^4/4]_0^2 = (3/8) \times (16/4) = 3/2$.
(b) $\mathrm{E}(X^2) = \int_0^2 3x^4/8 dx = (3/8) \times [x^5/5]_0^2 = (3/8) \times (32/5) = 12/5$.
(c) $\mathrm{E}(X^k) = \int_0^2 3x^{k+2}/8 dx = (3/8) \times [x^{k+3}/(k+3)]_0^2 = (3/8) \times (2^{k+3}/(k+3)) = 3 \times 2^k/(k+3)$.

4.2.9, $p_X(k) = P(X = k) = (1-p)^k.p$ for $k = 0, 1, 2, 3, 4......$
(a) $\mathrm{E}(X(X-1)) = \sum_{k=0}^\infty k(k-1)(1-p)^k.p = (1-p)^2.p.\sum_{k=0}^\infty k(k-1)(1-p)^{k-2}$
But $\sum_{k=0}^\infty k(k-1)y^{k-2} = \frac{d^2}{dy^2}\sum_{k=0}^\infty y^k = \frac{d^2}{dy^2}(1-y)^{-1} = \frac{d}{dy}(1-y)^{-2} = 2(1-y)^{-3}$.
So $\mathrm{E}(X(X-1)) = 2p^{-3}.(1-p)^2.p = 2(1-p)^2/p^2$.
(b) $\mathrm{E}(X^2) = \mathrm{E}(X(X-1) + X) = \mathrm{E}(X(X-1)) + \mathrm{E}(X) = 2(1-p)^2/p^2 + (1-p)/p = (1-p).(2-p)/p^2$.

(c) $\mathrm{E}((X - (1-p)/p)^2) = \mathrm{E}((X - \mathrm{E}(X))^2) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$
$= (2 - 3p + p^2)/p^2 - (1 - 2p + p^2)/p^2 = (1-p)/p^2$.

4.3.3, $f_X(x) = \lambda^2 x \exp(-\lambda x)$ on $x > 0$.
$\mathrm{E}(X) = \int_0^\infty \lambda^2 x^2 \exp(-\lambda x) dx = \int_0^\infty y^2 \exp(-y) dy/\lambda = 2!/\lambda = 2/\lambda$.
$\mathrm{E}(X^2) = \int_0^\infty \lambda^2 x^3 \exp(-\lambda x) dx = \int_0^\infty y^3 \exp(-y) dy/\lambda^2 = 3!/\lambda^2 = 6/\lambda^2$.
$\mathrm{var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = (6 - 2^2)/\lambda^2 = 2/\lambda^2$.

1.4.2 These are Bernoulli trials with $p = 0.85$.
(a) Number of Successes in 10 trials in $Bin(10, 0.85)$ and we want
$P(X = 8) + P(X = 9) + P(X = 10) = \binom{10}{2} 0.85^8 0.15^2 + \binom{10}{1} 0.85^9 0.15 + 0.85^{10} = 0.2759 + 0.3474 + 0.1969 = 0.8202$.
(b) First success not within three trials is Prob of 3 failures or $0.15^3 = 0.0034$.
(c) Between 3 rd and 4 th. success, is just as waiting time to first succcess.
The probability of at least two failures is $0.15^2 = 0.0225$.
(d) Exactly 6 successes in ten attempts, but first two are failures, is probability of 2 failues, then 6 successes in next 8 ($Bin(8, 0.85)$) so we have $0.15^2.\binom{8}{2} 0.85^6 0.15^2 = 0.00535$.