1. *(12 points: 3 each part)*
A continuous random variable X has probability density function (pdf)

$$\begin{aligned} f_X(x) &= \tfrac{1}{2}(x+1) \quad \text{if } -1 < x < 1, \\ &= 0 \qquad\qquad\quad \text{otherwise.} \end{aligned}$$

(a)

$$\begin{aligned} F_X(x) \;=\; P(X \le x) &= 0 && \text{if } x \le -1 \\ &= \int_{-1}^{x} \tfrac{1}{2}(y+1)\,dy \;=\; (x+1)^2/4 && \text{if } -1 < x < 1 \\ &= 1 && \text{if } x \ge 1 \end{aligned}$$

(b) $P(X > 0) \;=\; 1 - P(X \le 0) \;=\; 1 - F_X(0) \;=\; 1 \;-\; 1/4 \;=\; 3/4.$

(c) $P(X^2 < 1/4) \;=\; P(-\frac{1}{2} < X < \frac{1}{2}) \;=\; F_X(\frac{1}{2}) - F_X(-\frac{1}{2}) \;=\; 9/16 - 1/16 \;=\; 1/2.$

(d) $P(\exp(X) \;\le\; 1) \;=\; P(X \le \log(1) \;=\; 0) \;=\; F_X(0) \;=\; 1/4.$

2. *(12 points: 4 each part.)* Coffee mugs for a student union promotion are made in two colors, either blue or white. Since the white mugs are more popular, the manufacturer produces twice as many white mugs as blue. They are packed into crates randomly, so that each mug has probability 2/3 of being white and 1/3 of being blue, independently of other mugs in the crate.
The students decide to unpack mugs for display until they have at least 8 white mugs and at least 4 blue mugs.

(a) What is the probability the students unpack only 12 mugs in total?
This will be so if there are exactly 8 white and 4 blue in the first 12, in any order:
probability $\binom{12}{4}(2/3)^8(1/3)^4$.

(b) What is the probability the students unpack 8 white mugs and 8 blue mugs (16 in total)?
The 16th mug must be white, else they would have stopped sooner; so there are 7 white and 8 blue in the first 15, and then a white:
probability $\binom{15}{7}(2/3)^8(1/3)^8$.

(c) Suppose the students do unpack 16 mugs, 8 of each color. An assistant randomly puts 8 mugs on each of two display shelves. What is the probability there are 4 white mugs and 4 blue mugs on each shelf?
If it is so on one shelf, then it is so on the other. So select 8 mugs from 16, for the first shelf. Probability is

$$\frac{(\text{\# ways of picking 4 blue from 8})(\text{\# ways of picking 4 white from 8})}{(\text{\# ways of picking 8 from 16})} \;=\; \binom{8}{4}\binom{8}{4}/\binom{16}{8} \;=\; (8!/4!)^4/16!$$

3. *(12 points: 3 each part.)*
X, Y and Z are three random variables defined on the same sample space Ω.
It is known that $\mathrm{E}(X) = \mathrm{var}(X) = 1$, $\mathrm{E}(Y) = \mathrm{var}(Y) = 2$ and $\mathrm{E}(Z) = \mathrm{var}(Z) = 3$.
Additionally, the two random variables X and Y are independent.

(a) $\mathrm{E}(3X - 2Y + Z + 7) \;=\; 3\mathrm{E}(X) - 2\mathrm{E}(Y) + \mathrm{E}(Z) + 7 \;=\; 3 - 4 + 3 + 7 \;=\; 9.$

(b) $\mathrm{var}(3Z + 17) \;=\; 9\mathrm{var}(Z) \;=\; 27.$

(c) $\mathrm{var}(3X - 2Y + 7) \;=\; 9\mathrm{var}(X) + 4\mathrm{var}(Y) = 17$, using independence of X and Y.

(d) $\mathrm{E}(X(Y+2) + Z^2) \;=\; \mathrm{E}(XY + 2Y + Z^2) \;=\; \mathrm{E}(X).\mathrm{E}(Y) + 2\mathrm{E}(X) + \mathrm{var}(Z) + (\mathrm{E}(Z))^2 \;=\; 2+2+3+9 \;=\; 16$, using independence of X and Y for the first term, and the formula for variance.