Homework 1; Due 8.30 a.m. Wed Jan 14: Stat394 Final Exam

1. (a) Let T be the event of testing positive.

$P(T) \;=\; P(T \mid G)P(G) \;+\; P(T \mid G^c)P(G^c) \;=\; 0.95 \times .01 + 0.05 \times 0.99 = 0.059$

(b) $P(G \mid T) \;=\; P(T \mid G)P(G)/P(T) \;=\; 0.95 \times .01/0.059 = 0.161,$

$P(G^c \mid T) = 1 - P(G \mid T) = 1 - 0.161 = 0.839.$ (Or, $.99 \times 0.05/0.059 = 0.839$)

(c) $P(G \mid T^c) = P(T^c \mid G)P(G)/P(T^c) \;=\; .01 \times .05/(1 - 0.059) \;=\; .00053.$

2 (a) Note having H and testing positive for H are same thing, since there is no error in testing for H.

$P(H \mid T) = P(H \mid G)P(G \mid T) \;+\; P(H \mid G^c)P(G^c \mid T) \;=\; .161 \times (3/4) + .839 \times (1/4) = .3305$

(b) $P(H \cap G^c \cap T) \;=\; P(H \mid (G^c \cap T))P(G^c \cap T) \;=\; P(H \mid G^c)P(G^c \mid T)P(T) = .25 \times .839 \times .059 = 0.0124$

Or, $P(H \cap G^c \cap T) \;=\; P(H \mid (G^c \cap T))P(G^c \cap T) \;=\; P(H \mid G^c)P(T \mid G^c)P(G^c) = .25 \times 0,05 \times .99 = 0.0124.$

(c) $P(H \cap G \cap T^c) \;=\; P(H \mid G)P(G \mid T^c)P(T^c) = 0.75 \times .00053 \times (1 - 0.059) = 0.000375.$

Or $P(H \cap G \cap T^c) \;=\; P(H \mid G)P(T^c \mid G)P(G) = 0.75 \times 0.05 \times .01 \;=\; 0.000375.$

3. (a) $P(\text{bad}) \;=\; 1 - P(\text{good}) \;=\; 1 - P(0 \text{ faulty}) - P(1 \text{ faulty}) \;=\; 1 - (0.9)^{10} - 10 \times 0.1 \times 0.9^9 = 0.264$

(b) $P(2^{nd} \text{ bad is on trial } 9) \;=\; \binom{8}{1} \times .1^2 \times .9^7 = 0.0383.$ $\qquad P(\text{first 9 all good}) \;=\; 0.9^9 = 0.387.$

$P(\text{sold} \mid \text{exactly 9 tested}) \;=\; 0.387/(0.387 + 0.0383) = 0.91.$

(c) $P(\text{gadget sold} \mid \text{bad}) \;=\; P(\text{sold})P(\text{bad} \mid \text{sold})/P(\text{bad}) \;=$

$P(\text{first 5 good}) \times P(\text{2 or more bad in 2nd 5})/P(\text{bad}) \;=\; .59 \times .08/0.264 = 0.1788$

$P(\text{gadget discarded} \mid \text{good}) \;=\; P(\text{1 bad in first 5}) \times P(\text{0 in untested 5})/P(\text{good})$

$=\; .33 \times .59/(1 - 0.264) \;=\; 0.2645.$ (Note if more than 1 bad in first 5, gadget cannot be good)

4. (a) Let T be number of winners in any week. T is approx Poisson with mean $\mathrm{E}(T) \;=\; 40000/20000 \;=\; 2.$

$P(T > 2) \;=\; 1 - P(T = 0) - P(T = 1) - P(T = 2) \;=$

$1 - \exp(-2) - 2\exp(-2) - 2^2\exp(-2)/2 \;=\; 1 - 5\exp(-2) \;=\; .323.$

(b) Let X be the number of weeks (out of 208) with more than 2 winners. (Note $X \sim Bin(208, 0.323)$.)

$\mathrm{E}(X) \;=\; 208 \times 0.323 = 67.25,\ \mathrm{var}(X) \;=\; 208 \times 0.323 \times (1 - 0.323) \;=\; 45.5.$

(c) Approximately a standard Normal is $Z = (X - 67.25)/\sqrt{45.5}$ and we want $P(X > 59.5) = P(Z > -1.15)$.
From Table $P(Z < 1.15) = 0.875$ and, by symmetry, this is the required probability.

5. (a) $P(T > t) \;=\; \int_t^\infty f_T(s)ds \;=\; \int_t^\infty \lambda\exp(-\lambda s)ds \;=\; [-\exp(-\lambda s)]_t^\infty \;=\; \exp(-\lambda t)$

(b) Let the failure times be T_1 and T_2: $P(T_1 > t) = P(T_2 > t) = \exp(-t)$ (from (a)).

By independence: $P(T_1 > t \;\cap\; T_2 > t) = \exp(-t) \times \exp(-t) = \exp(-2t).$

Hence from (a), with $\lambda = 2$, (or by differentiating) we have the required pdf $2\exp(-2t)$.

(c) Suppose first component fails at s:
we want $P(T_2 > t + s \mid T_2 > s) = P(T_2 > (t + s))/P(T_2 > s) = \exp(-(t + s))/\exp(-s) = \exp(-t).$
(or, one can cite forgetting property of exponential here.)
Hence from (a), or by differentiating we get the required pdf $\exp(-t)$ on $0 < t < \infty$.

(d) After each repair, it runs in full mode for, on average, 0.5 weeks, and then as a copier for, on average, 1 week. So between each repair, it runs on average 1.5 weeks, and for 0.5 weeks in full mode: $0.5/1.5 = 1/3$.
OR: rate of failing when in full mode is twice the rate when running only as a copier, ...
(Note this is NOT a proof, just a reasonable explanation.)