## THE EMERGENCE OF INEQUALITY IN SMALL-SCALE SOCIETIES: SIMPLE SCENARIOS AND AGENT-BASED SIMULATIONS

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To appear in *Modeling Socioecological Systems*, ed. T. Kohler and S. van der Leeuw. Santa Fe: SAR Press.

[Revised May 2006]

A big family has cultivated fruitful soil; two little families nearby have thankless and rebellious fields. The two poor families have to serve the opulent family, or slaughter it; there is no difficulty in that. But one of the two indigent families offers its arms to the rich family in exchange for bread, while the other attacks and is defeated. The subservient family is the origin of the servants and the workmen; the beaten family is the origin of the slaves.

- Voltaire (from "Equality," Philosophical Dictionary, 1750)

Archaeological and ethnographic evidence indicates that human societies were egalitarian that is, lacking institutionalized political and economic inequality—for most of our species' history. Non-egalitarian societies first emerged in certain settled hunter-gatherer groups in the late Paleolithic and Mesolithic (10-15 kya), and began to proliferate with the spread of dense agricultural populations some 6-8 kya. Autonomous egalitarian societies of foragers and agriculturalists persisted in some regions into the ethnographic era, and some merged into expanding state societies only in the last few decades. Thus, egalitarian sociopolitical organization was remarkably stable in many areas for vast periods, yet eventually yielded to more hierarchical forms of social organization. One of the perennial grand questions of comparative social science is to understand how and why this happened.

While there is a large literature on the emergence of inequality, it contains very little formal theory or quantitative modeling. The research reported here is meant to address this gap. The simulations are not based on any particular society, but rather are designed to be general enough to apply to a wide range of possible cases. Drawing on game theory and evolutionary ecology, we use simple scenarios and agent-based simulation models to explore factors that might shape the emergence and stability of political-economic inequality in small-scale societies.

In the literature, such factors include population pressure; risk buffering and economic redistribution; information management; trade monopolies; spatially concentrated resources; military conflict; resource storage; control of production technology (e.g., irrigation); scalar

effects of increasing population density or community size; competitive feasting and gift-giving; and manipulation of esoteric knowledge or ideology (Hayden 1995). Thus, proposed causal variables range from environmental to demographic to economic to social to ideological. We cannot hope at this stage to cover this expansive waterfront, and have accordingly concentrated on two distinct scenarios. One, which we term (after Boone 1992) the "patron-client" scenario, focuses on economic exchange driven by initial differences in resource endowment (land productivity). The second, termed "managerial mutualism," envisages a social division of labor with the role of manager being to enforce adherence to collectively beneficial forms of production and distribution.

As with most theoretical efforts in evolutionary ecology, our primary aim is to construct models that reveal the outcomes that follow from a set of assumptions. Although our efforts here are exploratory and quite preliminary, we believe they offer some initial insights. Our models are predicated on the assumption that a limited number of asymmetries, such as differential control over productive resources, can explain the emergence of institutionalized inequality. They also draw on contemporary evolutionary theory in order to avoid the pitfalls of naïve functionalism and teleology. Our approach is not to deny any possibility of collectively beneficial outcomes or directionality to sociopolitical evolution, but rather to show if and how these might emerge from interaction of individual agency, social and demographic structure, and environmental constraints.

#### **Basic Simulation Structure**

Simulations were coded in C++ and run on Windows PCs. All results reported below involve runs of 2,000 rounds for each stated combination of parameter settings. Unless indicated otherwise, outcomes are reported for the last 10 periods averaged across a set of 100 runs per parameter combination (See the Appendix for a justification of parameter calibrations).

#### Environment

Agents inhabit patches arrayed in a square grid (e.g., 100 patches, 10 on a side). Patches are endowed with varying amounts of a generic resource, and patch richness (P) is assigned at the start of the simulation (See Tables 1 and 4 for a list of model parameters). Agents utilize this resource for survival and reproduction, and for economic activity (e.g., trade), but do not augment or deplete patch productivity; the resource of each patch is renewed at the start of each period.

#### Demography

In most of our simulations, each agent occupies a home patch continuously, and does not migrate. Agents reproduce at a rate proportional to their per-period income ( $\pi$ ), which is a function of their home patch's productivity, modified by any costs and benefits they accrue from social interactions (see below). To avoid ever-growing populations, a carrying capacity ceiling ( $K \le \Sigma P$ ) is imposed, and reproduction only occurs if there are undefended patches with sufficient untapped productivity to support additional agents. Offspring inherit the parent's strategy (subject to a probability [ $\mu$ ] per period of mutating to a randomly selected alternative), and populate available patches anywhere on the landscape (no population viscosity). Agents also face a mortality risk, with per-period mortality averaging one to five percent per period, assigned randomly but in inverse proportion to each agent's income ( $\pi$ ) in the preceding period.

#### Agent interaction

Depending on assigned strategy and the goals of the simulation analysis, agents may interact in a number of ways (sharing or not sharing patches, exchanging services and goods, cooperating in production). Further specification of agent interactions and payoffs are given below, as these vary between scenarios.

### **Patron-Client Scenario**

Many scholars, from Rousseau and Voltaire (see epigraph) to Marx and Engels, and beyond, have discussed the role that patron-client relations and private property (e.g., land ownership) may play in the emergence of institutionalized inequality in a previously egalitarian society. Our discussion here has been particularly influenced by the account in Boone (1992).

In our model of the patron-client scenario, some agents (Patrons) control patches with greater per capita resource endowments, and can trade access to these for services from less fortunate agents (Clients). Thus, heterogeneity in environmental productivity, and hence variation in property endowment, provides the initial opportunity for the emergence of inequality. Yet this is not sufficient, nor can this be glossed as "environmental determinism," since alternative strategies interacting with the same resource heterogeneity do not generate socioeconomic inequality. Some (Solo) agents simply defend richer patches for their exclusive use, while others (Doves) share any resources on their patch with other non-territorial agents (Doves or Clients). Thus, population density per patch (*n*) depends on both patch richness (*P*) and agent behavior. At the outset, all patches are seeded with a single occupant playing the Dove strategy; by making all-Dove the initial state, we examine when inequality can evolve from an egalitarian starting point.

#### Patch defense

Resource control (via territorial patch defense) is critical to the patron-client scenario (Boone 1992). Territorial agents (Solo and Patron strategies) pay a cost (*d*) to defend sole occupancy of their patch, regardless of patch productivity (as long as  $P \ge 1-d$ ). A territorial agent cannot colonize a poor patch (P = 1). Other strategies (Dove and Client) do not defend, and will thus equally share the productivity of their patch (but not other income, i.e., Client exchange earnings) with co-resident non-territorial types. For patch-sharing types, P/n must be  $\ge 1$  (though a sufficient amount of exchange earnings for Client types could relax this constraint, a possibility we have not modeled). At equilibrium, we thus expect that population density per patch (*n*) will match patch productivity (*P*) for non-territorial types (i.e., approach an ideal-free distribution), but be lower for territorial types (where  $n \le 1$  regardless of *P*). Thus territoriality should lower total population (*N*) and social efficiency ( $\Sigma\pi$ ).

#### Political economy

Solo agents do not interact with others, while Doves simply coexist with other Doves or Clients in richer (P > 1) patches. Thus, the only economic exchanges are between Clients and Patrons. We assume that Patrons can have multiple Clients (but not vice versa); it might prove interesting to relax this assumption. Patrons maintain exclusive control of resources on their patch, but are willing to exchange some share ( $\kappa$ ) of *P* with Clients for any profitable return ( $\tau$ ). Clients are willing to expend labor costs ( $\lambda$ ) in exchange for a profitable wage  $\kappa$  from the Patron

offering the best deal. Thus, exchanges are subject to the condition that  $\tau > \kappa > \lambda$ . In the initial simulation, the values of these exchange variables are set exogenously, and we assume that any Patron-Client exchange divides the surplus equally (i.e., that  $\kappa = (\tau + \lambda)/2$ ).<sup>1</sup> Strategies and payoffs are listed in Table 2.

*Simulation results.* Due to stochasticity in initial conditions, few if any parameter settings produce invariable outcomes (Figure 1). In particular, given the initial seeding of all patches with a single Dove agent (Figure 2), much depends on the chance that territorial (Patron or Solo) mutants will replace solitary Doves inhabiting richer patches relatively early in a simulation run, and thus gain a substantial toehold in the population. This stochastic element aside, the basic dynamical behavior of this simulation is easily summarized. We concentrate here on factors that favor a stable patron-client system (i.e., one in which Patrons and Clients are numerically dominant and resist replacement by other types.

Under default parameter values (see Table 1) non-territorial strategies dominate, split equally between Dove and Client types, and Solo and Patron types are about equally represented in the remaining patches. However, a stable patron-client regime emerges in about one third of all runs, and takes over the population about 10 percent of the time (Table 3).

The hiring returns parameter ( $\tau$ ) has the strongest effect on emergence and stability of a patron-client regime. This is quite intuitive, since the value of this parameter strongly affects the payoffs of both Patrons (who receive  $\tau$  from each Client with whom they interact) and Clients (who receive a wage  $\kappa$  proportional to  $\tau$ ). When returns are high ( $\tau = .6$ , twice the default setting), virtually all patches are occupied by Clients and Patrons in about a 5:1 ratio; over 90 percent of runs end with almost no other strategies present in the population (Table 3). Conversely, low returns ( $\tau = .1$ ) are very unfavorable for Clients (who then cannot do as well as Doves and thus remain a tiny minority of the population). Intermediate hiring returns ( $\tau = .4$  or .5) are quite favorable for the patron-client regime, with the ratio of Patrons to Clients reaching about 3:1, and half to three-quarters of the runs end in with these two strategies constituting over 90 percent of the population (Table 3). The same results are attained with less favorable hiring returns but lower labor costs (Table 3).

Demographic parameters have a strong effect on the relative success of territorial and nonterritorial strategies. When mortality is high or reproductive rate low, the initial (all-Dove) population expands slowly so that Solo and Patron agents are able to spread and control rich patches, effectively keeping Dove and Client numbers low at equilibrium. Conversely, low mortality or high reproductive rate allows Doves to proliferate rapidly, and territorial agents are locked out (with Clients arising in modest numbers through mutation and drift). Increased mutation rates are favorable to the spread of Client and Patron strategies, but only because this retards the initial proliferation of Doves.

As anticipated by Boone (1992), environmental heterogeneity is critical, as Patrons capitalize on their relatively rich patch endowments to participate in exchanges with Clients. Most of our simulations feature a random uniform distribution of patch richness (*P* varying from 1 to 5), with territorial agents (Solo or Patron) restricted to patches with  $P \ge 2$ . Since territorial agents do not proliferate within their patches, their average patch resources equal 3.4 (the mean of P = 2 to 5, minus defense cost), whereas non-territorial agents proliferate to the point where average per capita patch resources approach 1. In simulations where patch richness is set uniformly high (P = 5) territorial strategies are very rare (presumably due to the rapid proliferation of non-territorial agents), whereas in uniformly poor environments (P = 2) the

opposite results. In both cases, homogenous environments effectively prevent the emergence of a Patron-Client system (Table 3).

The remaining parameters have only modest effects on strategy success. As might be expected, low labor costs ( $\lambda = .1$ ) are favorable to establishment of a patron-client regime, while high labor costs ( $\lambda = .4$ ) keep Client frequencies very low (Table 3). Territory defense costs, paid by all Solo and Patron agents, have surprisingly little effect on simulation outcomes; high costs (d = .4) hurt Solo more than Patron types, presumably because the latter can recoup some of these costs via exchanges with Clients, but as they have no direct effect on Clients the net effect on emergence of patron-client regimes is insignificant.

#### **Managerial Mutualism**

Many scenarios regarding the origins of inequality posit that hierarchy emerges from mutualistic arrangements involving specialized socioeconomic roles (e.g., Johnson 1982; Keeley 1988; Service 1975). These accounts envision a small number of managers (chiefs, priests, etc.) who provide benefits to the members of their social group by coordinating important aspects of production, distribution, or information flow. In light of established theory in evolutionary ecology and collective action, some of these scenarios are more plausible than others.

Our analysis of what we term "managerial mutualism" focuses on the whether managers who help solve collective action problems within a social group can instigate institutionalized inequality in a previously egalitarian system. Specifically, we seek to address the factors that favor the emergence of such a system of managerial mutualism, and the conditions under which those who manage collective action accrue above-average gains.

#### Demography

Agents reside in social groups of size *n*, which can reach some local maximum size *n*'. A proportion *p* of the *n* agents play Cooperate and the remainder play Defect, except for groups that also have one agent playing the Manager strategy (see below for strategy specifications). Agents reproduce at a rate proportional to their per-period income ( $\pi$ ), which is a function of their base income augmented by any gains (or losses) from collective action and its related enforcement costs (as explained shortly). Offspring can colonize empty or partially saturated (*n* < *n*') patches, with no viscosity or migration cost. Agents also face a mortality risk averaging one to five percent per period, assigned randomly but in direct proportion to each agent's income in the preceding period (see Table 4 for a list of all parameters and their assigned values). Offspring inherit the parent's strategy (subject to a probability  $\mu$  of mutating to a randomly selected alternative). However, the Manager strategy is fixed at  $\leq 1$  per group and is thus not subject to within-group proliferation. Accordingly, offspring of Managers who cannot colonize another (no-Manager) group retain their Manager "genotype" but play the Cooperator strategy. If the resident Manager dies, and any such "phenotypic Cooperators" are resident in a group, one is randomly selected to become next Manager.

#### **Collective** Action

Agents interact once per period in a public goods game, which we envision as a form of group production with gains to cooperation, such as running an irrigation system or collectively harvesting prey. Cooperators contribute to production of this public good at a personal cost (c); both Cooperators and Defectors (who do not contribute to group production) obtain an equal

share (pG/n) of this production. To prevent free riders from completely unraveling this cooperation, agents must be monitored and punished if they fail to contribute. This enforcement is costly to the enforcers, creating a second-order collective action problem. We can envision various possible outcomes: a) collective action fails and agents obtain only their base income; b) some level of collective action is achieved, and Cooperators pay the enforcement costs; or c) one agent assumes the specialized role of Manager and pays all enforcement costs in return for a profitable managerial fee (paid by Cooperators). This last outcome is the one of primary interest here.

## Payoffs

An agent's net income per period  $(\pi)$  is a function of that agent's strategy and interactions; we do not include environmental heterogeneity directly in this model, though variation in base income and collective production (see below) can be viewed as environmentally constrained. We assume that Defectors neither contribute to the costs of public good production, nor to monitoring and punishing free riding. Each agent has a base income *B* that is independent of agent type and of shares from collective action; since this just adds a constant to every payoff, it has no effect on relative payoffs and we omit it from the equations below. If there is no Manager present in a local group, then for each Cooperator, payoffs are

 $\pi_C = pG/n - c - nm - n(1-p)e$ 

and for each defector

 $\pi_D = pG/n - nps$ 

where *n* is the number of agents in the local group (not counting the  $\leq 1$  agent who acts as Manager), *p* is the proportion of *n* agents who Cooperate and 1-*p* the proportion who Defect, *G* is the maximum possible aggregate public good production, *c* is the cost per Cooperator of producing *G*, *m* is the cost per Cooperator of monitoring all other agents, *e* is the cost per agent of punishing one Defector, and *s* is the cost of being punished by one Cooperator. If *G*/*n*<*s*, there is no incentive to defect from cooperative production, and (1-p)e tends toward zero as Defectors become rare; however, this requires that the second-order problem of monitoring and enforcement has been solved, which is in fact the key problem on which the managerial mutualism scenario focuses.

If the local group includes a Manager, so that group size is n+1, then payoffs are as follows:

$$\pi_{C^*} = pG/n - c - \gamma$$
  

$$\pi_{D^*} = pG/n - S$$
  

$$\pi_M = n\{p\gamma - M - (1-p)E\}$$

where  $\gamma$  is the "management fee" that each Cooperator pays the Manager, *M* is the cost the Manager pays to monitor cooperation, *E* is the cost the Manager pays to punish a Defector, and *S* is the cost a Defector suffers when punished by a Manager. We assume that Managers are at least as good at monitoring as are Cooperators (i.e., there can be gains to specialization), so that  $M \leq m$ . We also assume that Managers can impose a larger punishment on any Defector, so that S >> s. Note that Managers are enforcement specialists, and do not directly produce nor consume the collective good; relaxing this last assumption would not alter the qualitative results, but would significantly complicate the algebraic representation.

## Effect of Parameter Values on Equilibria

Since we model this scenario in explicit game-theoretical form, some analytical conclusions (independent of simulation results) can be derived. Specifically, we deduce the following:

- 1. The collective action problem can be solved without a Manager if the proportion of cooperators (*p*) is sufficiently high. With the default values of other parameters, the critical value of p = .636, which is the value at which the payoffs to Defector and Cooperator are equal (assuming no Manager is present). This value of *p* defines an unstable equilibrium point: if the initial p > .636, then p goes to 1 (i.e., the all-Cooperator equilibrium), and if the initial p < .636, then p goes to 0 (i.e., the all-Defector equilibrium), as illustrated in Figure 3a.
- 2. Managerial mutualism is a Nash equilibrium if  $\pi_M > \pi_{C^*}$  and  $\pi_M > \pi_{D^*}$  (see above for payoff abbreviations).
- 3. Under the right initial conditions,  $\pi_{C^*} > \pi_{D^*}$  even when cooperation is relatively rare (Figure 3b); thus Cooperate can invade a group of Defectors as long as the Manager strategy arises simultaneously.
- 4. Group size (*n*) affects the relative payoffs of Cooperators and Managers, and hence prospects for managerial mutualism; specifically, *n* must be large enough to provide an adequate incentive to Manager but not so large that the value of the collective good is shared among too many cooperators.
- 5. The aggregate value of the collective good *G* has no effect on relative payoffs to Cooperate or Defect, but if *G* is too large then the payoff to Manager is lower than that to Cooperator, and the managerial equilibrium is less likely.
- 6. The labor cost of producing the collective good (*c*) cannot be too high or Cooperators do worse than Defectors, even with a Manager present.
- 7. If the monitoring cost (*M*) or enforcement cost (*E*) experienced by a Manager get too large, the Manager strategy cannot persist.
- 8. Managerial mutualism requires an sufficiently high value for *S*, a Defector's cost of being punished; if *S* is too small then Defectors outcompete Cooperators even when a Manager is present. We have adopted a default value for *S* that is six times the cost a Manager pays to punish a Defector, but with default values of other parameters *S* can drop as low as 1 even if half the group is Defectors. However, the emergence of the managerial regime requires that *S* be not much less than the default value of 3.
- 9. The management fee ( $\gamma$ ) must be in a relatively narrow range to maintain the managerial equilibrium. If it is too low then Managers cannot prosper; too high and Cooperators are better off without Managers.

## Simulation results

As shown in Table 5, the simulation model closely matches the analytical results just described. A system of managerial mutualism emerges readily (nearly 90 percent of the time) under default parameter values, and is even more likely if production costs for the collective good are lowered. The MM equilibrium is less likely to evolve if production costs are high, management fees are too high or too low, the value of the collective good strays too far from the default value, or costs to managers of monitoring or punishing defectors are high. Demographic parameters (mortality, fertility, and group size) also have a predictable effect on the MM regime. The punishment cost borne by defectors is of course critical as well, as discussed above.

Our simulation results indicate that the present model is very sensitive to initial conditions. In particular, the Manager strategy must be present in sufficient numbers in the population (roughly one Manager per group) to allow the MM equilibrium to evolve. This, plus the narrow value ranges required on several parameters (see above), presents a more difficult route to the emergence of inequality than the Patron-Client model. This finding could be an artifact of model structures or assumed parameter values, but for various reasons we suspect it reflects a realistic difference between systems based on dyadic mutualism (as in the Patron-Client scenario) and those based on collective action (as in managerial mutualism).

#### Future Refinements

Our model assumes that Defect and Cooperate are fixed strategies. If we were to make Defect a conditional strategy, so that Defectors only shirk if the expected cost of being punished is lower than cost of contributing to public good (s < c), then realized enforcement and punishment costs approach zero, and payoffs change accordingly. But in the long run, absence of defection will favor evolution of variants who don't bother to monitor and punish (Boyd et al. 2003). It is possible that the Manager strategy will suffice to prevent this subversion, but further modeling would be required to examine this.<sup>2</sup>

The fee that Cooperators pay to the Manager ( $\gamma$ ) could be made proportional to group productivity (i.e.,  $\gamma = \text{fn}(G)$ ), rather than a flat per-producer amount. Even more interesting would be to allow  $\gamma$  to evolve (via random mutation) or to vary strategically in a population already consisting of stable groups with managers. In such a context, we could expect the Managers to benefit from setting the management fee as high as producers would tolerate, right up to the point where the Cooperators would do better without a Manager.<sup>3</sup> This is precisely the type of dynamics envisioned in many models of "reproductive skew" (Reeve and Emlen 2000; Verhrencamp 1983). Cooperators would then have the choice of a) staying in the group while paying the fee set by the manager, b) joining another group with a Manager offering better terms, or c) joining a group without a Manager.

It would also be instructive to model competition between Managers. This could happen either between groups (the Manager in one group trying to offer a better deal to draw producers in from other groups), or within groups (someone trying to usurp the existing Manager by offering a better deal).

Some of these options open the door to analyzing potentially interesting effects of population structure and migration, including multi-level selection. Of course, such enrichment would come at the cost of a corresponding increase in complexity of model design and interpretation.

#### **Conclusions and Prospects**

In the history of social thought, accounts of the rise of inequality tend to sort into two categories: those that emphasize the benefits that hierarchy brings to all (what a biologist would call mutualism), and those that emphasize exploitation or coercion by one segment of society (elites) against the interests of the remaining members (commoners, producers, slaves, etc.). Of course, many scenarios combine both elements, but this mutualism/exploitation contrast is a useful one, particularly if viewed as a continuum rather than a dichotomy. Both of the scenarios analyzed in this paper fall towards the mutualistic end of this continuum, although this is less true of the patron-client scenario.

Despite the mutualistic nature of the underlying interactions, both scenarios are capable of producing marked and stable inequality in income. This inequality averages about 2.2:1 for the Patron:Client model examined here, and about 1.4:1 for the Manager:Cooperator model. The evolutionary dynamic translates income into enhanced survival and fertility, and helps determine the equilibrium frequency of different strategies. But structural aspects of the strategies and their

interactions are equally important, and in fact mean that the wealthier agents in each scenario (Patrons and Managers, respectively) are actually less numerous than their poorer counterparts. In the case of Patrons, this is because of their territorial exclusivity, which gives them a richer resource base than Clients (and thus the wherewithal to exchange surplus resources for Client services) but also lowers their population density. In the case of Managers, role specialization limits Manager frequency to one per local group—somewhat artificial yet plausible in making producers more numerous than administrators—although our model allows Manager offspring to proliferate and phenotypically display the Cooperate strategy if no Manager openings are available.

As Clark and Blake (1994:17) have argued, "explanations of the origins of institutionalized social inequality and political privilege must resolve the central paradox of political life—why people cooperate with their own subordination and exploitation in non-coercive circumstances." We propose that a limited number of asymmetries can explain most cases of emergence of institutionalized inequality. These might include asymmetries in control over productive resources, control over external trade, differential military ability (and resultant booty and slaves), or control of socially significant information. As our models demonstrate, these asymmetries need not be employed coercively, as long as they are economically defensible (*sensu* Brown 1964) and can provide an advantage in bargaining power sufficient to allow the concentration of wealth and/or power in the hands of a segment of the social group or polity. Our modeling indicates that such asymmetries can be self-reinforcing, and thus quite stable to moderate perturbations over time. Because most of the social transactions based on them are mutualistic rather than coercive, we suggest that such systems are likely to be more stable than the stratified social systems (e.g., nation states) that eventually succeed them.

Clearly there are many directions for future work. Besides further elaboration of these two models, along lines suggested above, new scenarios that feature other determinants of the emergence of inequality could be developed using the same methods of evolutionary game theory and agent-based simulation we have utilized here. Plausible candidates for such additional determinants include inter-group trade (the germs of which are present in the patron-client model), warfare, and multi-player alliances. In addition, current and future scenarios could explore the effects of adding more dynamism to agent-environment interactions (such as resource depletion and enhancement), and more population structure (which might create opportunities for multi-level selection). We nevertheless stress the value of starting with the simplest interesting models possible for any given scenario, in order to better understand the results of simulations as well as to articulate these with analytical insights such as those offered by game theory.

Finally, there is the matter of relating models to empirical data. Unlike most of the research reported in this volume, our goal was not to analyze a particular socio-ecological system found in a particular time and place; accordingly, we emphasized generality and simplicity of model structure and dynamics. Nevertheless, our ultimate goal is to understand patterns found in the real world. This will require the generation of testable hypotheses from these or similar models, and empirical analyses to determine the extent to which they might help explain the emergence of inequality in actual transegalitarian societies.

*Acknowledgments*. EAS thanks Sam Bowles for hosting his stay at the Santa Fe Institute in 2004 where this work began, and Jim Boone for much inspiration and helpful discussion over the years. Many thanks to Rob Boyd, James Kitts, Fraser Neiman, and Lore Ruttan for comments on earlier versions of the paper, and the National Science Foundation (BCS-0314284) and SFI for support.

#### **Appendix: Parameter Specification and Calibration**

As noted in the introduction, the models in this chapter are not intended to realistically reflect the characteristics of any particular society, but rather to be as simple and general as possible so as to provide insights into a few important causal variables that might be important in many transegalitarian societies. Nevertheless, the choice of parameter values deserves some justification. (Note: the abbreviations PC and MM below refer to patron-client and managerial mutualism models, respectively.)

#### Demographic Parameters

These determine the size of local groups, basic demographic rates of agent replication and mortality, and the probability of strategy mutation. No attempt is made here to model real demographic phenomena, and the model populations have no sexual reproduction, no age structure, and no households.

Run length: Our simulations utilized runs of 2,000 periods, with each period characterized by some probability of reproduction and mortality (as a function of payoffs in the preceding period) as well as by social and economic processes that determine agent payoffs. For both MM and PC, 2,000 periods seemed long enough to allow full dynamical behavior (in cases where parameter settings led to cycling) or equilibrium (in other cases). However, this was judged by inspection rather than by systematic experiment.

Agents in a patch or group (n): In the PC model, the size of local groups co-residing in patches is limited by 1) agent strategy (some types are territorial and thus are the sole inhabitants of their patches) and 2) patch richness (non-territorial types proliferate on or migrate into patches until they reach local carrying capacity). In the MM model, groups of size n share collective goods and engage in other interactions. In both cases, n is kept deliberately small (5 to 15 individuals) in order to minimize complexity and (in the case of MM) make collective production more feasible.

Total population size (*N*): No global population limit is imposed, but an equilibrium population emerges from constraints on local group size (*n*), as just described. The resulting values of *N* range from about 100 to 300 for the PC model, which is quite realistic for a regional population in small-scale societies of foragers and horticulturalists. For the MM model, *N* is typically around 1,000, but all interaction is local (i.e., in groups of size n = 8 to 15 agents), although newly born agents can and do migrate outside their natal group.

Reproductive rate: The default rate was set to .025 in both models, and varied over a range from .01 to .05. Since our simulated populations have no sex or age structure, it is difficult to compare this rate to demographic measures in real populations. Perhaps the least problematic comparison is our net reproductive rate (reproduction minus mortality) with population growth rates. With model mortality rates generally set below reproductive rates, this net rate was typically about .01 or less, equivalent to  $\leq 1$  percent population growth rate per period, a demographically reasonable amount if we think of a simulation period as equal to one year. Note that this net growth rate was highest when habitats were unsaturated (i.e., at the start of a simulation run), and declined to near zero at equilibrium, which is again a realistic pattern.

Mortality rate: As with reproductive rate, mortality rate was a function of an agent's payoff in the preceding period (an inverse function in the case of mortality). We constrained this to cover a moderate range of rates, typically from .01 (highest payoff) to .05 (lowest payoff). Although real populations have higher rates than this in the youngest and oldest age classes,

given the lack of age structure and that reproduction was allowed for all agents, the rates utilized here appear reasonable.

Mutation rate ( $\mu$ ): The default probability (per period) of randomly changing one's strategy to another strategy was .01 for both models. This value is much higher that what we might expect for genetically transmitted variants, but probably lower than is typical for culturally transmitted ones. Varying mutation rate from .001 to .05 had little or no effect on simulation outcomes.

Proportion of Cooperators in local group (p): This parameter indicates the proportion of agents in local group who are Cooperators (MM model), and is a multiplier for several other parameters that enter into the payoffs for each agent type. The default initial value of p was set at .33 (i.e., a uniform distribution of the three strategy types) in most runs.

#### Income Variables

Patch productivity ( $P_j$ ): Productivity of the  $j^{th}$  patch type is set exogenously in the PC model. For most simulations, there were five discrete types with five possible values (1, 2, 3, 4, 5), assigned randomly but with equal probability. Environmental heterogeneity is widespread in nature, and many scenarios of the emergence of inequality give it some role; we thus examined a variant with no heterogeneity (i.e., setting all patches to a single type ( $P_j = 2$  or  $P_j = 5$ ), and another variant with a spatial pattern of increasing productivity from one corner of the lattice to the opposite corner. We intend to investigate a future variant where  $P_j$  will fluctuate stochastically over various ranges during runs.

Net income  $(\pi_i)$ : Net income of the  $i^{\text{th}}$  agent is a function of the payoff equations for each strategy, as specified in the text. It is calculated for each period, and then fed to routines calculating demographic parameters (reproduction and mortality), as noted above.

Base income (B): Base income is the portion of net income that each agent obtains exclusive of gains from collective action or managerial role (MM model). Since we assume it is equal for all types in any given run, it has no effect on relative payoffs, but it is convenient to avoid mechanical problems that might otherwise ensue from negative payoffs. We set the default value equal to the gain from collective production if all agents in the local group were Cooperators, and varied it to ensure that it had no effect on strategy proliferation.

#### Interaction Variables for Patron-client Model

Territory defense cost (*d*): Territorial strategies (Solo and Patron) are assessed a cost *d* (per period) for maintaining exclusive control over the resources of their patch, which means that agents playing territorial strategies can only inhabit patches with productivity  $\geq 1 + d$ . This seems reasonable, since such exclusive control provides benefits (patch resources) that would otherwise go to other agents, and in real social systems (human or non-human) such resource defense is contested and thus costly. In our model, however, we do not explicitly model contests over resources, and thus one can consider *d* a display cost that advertises the residents' willingness to fight to repel intruders. We assume in effect that territorial residents always win such contests, and that non-territorial strategies (Dove and Client) share the resources of their home patch equally without transaction costs. We have rather arbitrarily set the default value of *d* at .1; varying this from zero to four times the default value had almost no effect on the emergence and stability of the PC regime.

Client's labor cost for services to Patron ( $\lambda$ ): This measures the cost to a Client of providing services to a Patron (per period). We have assigned  $\lambda$  a default value of .2; doubling this has a

pronounced negative effect on the emergence of the PC regime, while halving it has a clear positive effect.

Patron's return from a Client's labor ( $\tau$ ): In our model, a Patron hires one or more Clients in order to obtain some net benefit from each exchange. Each Client's labor provides a gross benefit  $\tau$  to the Patron, who pays the Client a wage  $\kappa$ , and thus receives a net benefit of  $\tau$ - $\kappa$ . The default value of  $\tau$  is .3, meeting the requirement that  $\tau > \kappa$  (i.e., that the exchange is profitable for the Patron). Because  $\tau$  enters into the payoff equations of both Patrons (directly) and Clients (indirectly, via its contribution to a Client's wages  $\kappa$ , as discussed below),  $\tau$  has a strong effect on the PC equilibrium. At the default value of  $\tau = .3$ , 24 percent of runs end with a clear PC equilibrium, whereas at  $\tau = .1$  none do and at  $\tau = .6$  virtually all (98 percent) do.

Patron's payment for Client services ( $\kappa$ ): Without coercion or deception, we should not expect an agent to engage in an exchange without profit. In the present case, this means that a Client will not provide services to a Patron unless wages exceed labor costs ( $\kappa > \lambda$ ). Since profitability for the Patron requires that  $\tau > \kappa$ , these dual constraints allow  $\kappa$  to lie anywhere between  $\tau$  and  $\lambda$ , which at the default values means  $.2 < \kappa < .3$ . While future analysis should allow  $\kappa$  to be set by either evolutionary dynamics or agent bargaining power, here we have assumed the most neutral option of an equal division of the surplus by setting  $\kappa = (\tau + \lambda)/2$ .

#### Interaction Variables for Managerial Mutualism Model

Maximum aggregate gains from collective action (*G*): In the MM model, Cooperators engage in collective action to produce a collective good, which is divided equally among the *n* agents in the group (except the Manager). The amount of collective good produced is a function of the number of Cooperators, with a maximum value of *G*. Thus, any Cooperator's or Defector's share of the collective good is pG/n. The default value of *G* is 50, so that in a group of all Cooperators the per capita share is 5, equal to the default value of base income (*B*).

Cost of producing collective good (c): The labor cost of producing the collective good, per Cooperator. The assigned default value is 1; a high value depresses the equilibrium frequency of Cooperators and Managers, whereas a low value increases these.

Cooperator's cost of monitoring (m): We assume that monitoring to detect instances of defection is independent of number of Defectors in the group, but is proportional to group size (n). The default value is set at 0.1n, unless a Manager is present in which case it falls to zero.

Manager's cost of monitoring (M): We set this at 0.1 as well, thus ignoring possible efficiency gains to specialization. As with monitoring by Cooperators, this is proportional to group size.

Cooperator's cost of punishing (*e*): With no Manager present, a Cooperator pays 0.5 for each Defector in the group. The total costs of enforcement per Cooperator are (1-p)e, and thus go down as *p* (the frequency of Cooperators in the group) goes up; conversely, the cost of being punished per Defector is *ps*, and thus goes up with Cooperator frequency. With a Manager present, Cooperators pay no enforcement cost, and Defectors are punished only by the Manager.

Manager's cost of punishing (E): A Manager pays 0.5 for each Defector, and the total enforcement cost to a Manager goes down with p (simply because they are less Defectors to punish).

Cost to Defector of being punished (S or s): In groups without a Manager, punishment is done by each Cooperator in the group, and the realized cost to each Defector of being punished (ps) increases with number of Cooperators in the group. In groups with a Manager this cost does not vary, and Defectors are punished by the resident Manager. Punishment cost ratio: The ratio of a Defector's cost of being punished to cost of punishing a Defector is widely assumed to be >>1, on the grounds that punishers can choose circumstances to their advantage (e.g., employ surprise), and utilize technology to punish at a distance (Bingham 1999). In our model this ratio differs depending on whether it is a Cooperator or a Manager who is doing the punishing. With no Manager present, the punishment cost ratio is a function of the number of Cooperators present, reaching 6:1 when Cooperators and Defectors are equally frequent. When a group has a Manager, the ratio is 3/.5, or 6:1. Thus, we assume that a Manager is both more efficient and more effective than a single Cooperator in punishing each Defector, on the basis of role specialization (and possibly self-selection, as per the signaling model in Gintis, Smith, and Bowles 2001). Note that Boyd et al. (2003) assume a punishment cost ratio of 4:1, which is similar to our ratio for Manager-Defector interactions. Even if we allow the punishment efficiency to be as high for Cooperators as for Managers, it has no effect on the emergence of the MM system.

Managerial fee ( $\gamma$ ): Fee paid to the Manager by each Cooperator in the local group. The default value is 1, and the MM equilibrium is quite sensitive to this value (Table 5). If  $\gamma$  is too low, Managers cannot survive, whereas if it is too high Cooperators suffer depressed payoffs and hence lower success.

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### Notes

<sup>1</sup> In later versions, we intend to let values of  $\kappa$  and  $\tau$  vary, either by mutation or strategically. If Clients can search globally for the Patron offering the best deal,  $\kappa$  should assume a uniform value in the population. A more interesting variant would specify a diminishing marginal value of Clients for any Patron; this would entail a decline in  $\tau$  as a Patron gains additional Clients, and a corresponding decline in the value of  $\kappa$  a Patron will offer existing or new Clients.

<sup>2</sup> If Defectors are effectively eliminated, Manager variants who extract a managerial fee without providing enforcement benefits would be behaviorally indistinguishable from the original variant and could perhaps drift in to the population (Rob Boyd, personal communication, September 21, 2005).

<sup>3</sup> We are indebted to James Boone (personal communication, May 13, 2004) for making this point.

Symbol	Definition	How set	<b>Default</b> <sup>a</sup>	Min.	Max.
d	Territory defense cost	Exogenously fixed	.1	.01	.5
μ	Strategy mutation probability	Exogenously fixed	.01	.001	.05
n	Number of agents on patch <i>j</i>	Demographic	variable	0	5
N	Total number of agents	Demographic	variable	~80	~230
$P_{j}$	Productivity of the <i>j</i> th patch	Exogenously fixed	random	1	5
$\pi_i$	Net income of the <i>i</i> th agent	Agent behavior	variable	1	var.
	Reproductive rate	Function of $\pi_i$	.025	.1	.5
	Mortality rate	Function of $\pi_i$	.01 to .05	.02 <sup>b</sup>	.1 <sup>b</sup>
τ	Patron's return from a Client's labor	Exogenously fixed	.3	.1	.5
λ	Client's labor cost	Exogenously fixed	.2	.1	.5
κ	Patron's payment for Client services	Function of $\kappa$ and $\lambda$	$(\tau + \lambda)/2$	.15	.4

## Table 1. Parameters of the Patron-client Scenario.

<sup>a</sup> Initial value; all values are per period unless otherwise noted <sup>b</sup> Minimum and maximum of the upper end of the mortality range (see text)

Strategy	Behavior	<b>Payoff</b> <sup>a</sup>
Dove	Occupy any patch with $P/n \ge 1$ , share with others	$P_j/n$
Client	Same as Dove, plus exchange labor for resources with Patron	$P_i/n + \kappa - \lambda$
Patron	Defend exclusive control of patch, exchange resources for services from Clients	$P_j - d + \Sigma(\tau - \kappa)$
Solo	Defend exclusive control of patch, no exchanges	$P_j - d$

## Table 2. Strategies and Payoffs in the Patron-client Scenario.

<sup>a</sup> See Table 1 for definition of variables

		Mean terminal frequency:						
Parameter	Value	Dove	Solo	Client	Patron	90% <sup>a</sup>	Comments	
All default values	(see Table 1)	.29	.19	.29	.23	.10	Stable PC regime emerges in about 1/3 of runs	
Mutation rate ( <i>m</i> )	High (.05)	.16	.11	.46	.27	.10	PC regime dominates somewhat more often	
Defense cost (d)	High (.4) None (0)	.33 .31	.12 .21	.33 .29	.23 .19	.13 .09	Little effect compared to default settings Same	
Hiring returns (τ)	Low (.1) Moderate (.4) Higher (.5) High (.6)	.52 .14 .07 .02	.21 .09 .07 .02	.06 .54 .66 .81	.22 .23 .20 .16	0 .49 .76 .94	Client strategy does very poorly More favorable for PC regime than default Even more favorable Very stable and consistent PC regime	
Labor cost $(\lambda)$	High (.4) Low (.1)	.52 .14	.21 .09	.06 .54	.22 .23	0 .49	Client strategy does very poorly More favorable for PC regime than default	
Increased hiring returns + decreased labor costs	$(\tau = .4, \lambda = .2)$	.07	.07	.66	.20	.76	PC regime usually dominates	
Mortality curve	Steep (max = .1) Shallow (max = .02)	.06 .89	.42 0	.05 0.11	.47 0	.18 0	Highly variable outcomes; generally favors territorial strategies (Solo & Patron) Doves spread rapidly, so no PC regime	
Reproductive rate	High (.05) Low (.01)	.75 .00	0 .44	.25 .00	0 .55	0 0	Territorial strategies cannot get established Territorial strategies completely dominate	
Lattice size	Larger (20x20)	.15	.18	.37	.30	.05	Slightly more favorable to PC regime	
Patch richness	Uniformly high $(P_j = 5)$ Uniformly low $(P_j = 2)$	.51 .04	.01 .33	.46 .05	.02 .57	0 0	Territorial strategies very rare Territorial strategies dominate	

 Table 3. Patron-client Simulation Outcomes under Different Parameter Settings (Means = Last 10 Periods over 100 Runs, 2000 Periods per Run).

<sup>a</sup> Proportion of runs in which >90 percent of agents are playing Patron or Client strategies by the last 10 periods

Symbol	Definition	How set	Default Value <sup>a</sup>	Min.	Max.
μ	Strategy mutation rate	Exogenously fixed	.01	.001	.05
n	Maximum number of agents in local group	Exogenously fixed	10	5	15
	Reproductive rate	Exogenous range, fn( $\pi_i$ )	.025	.02	.05
	Mortality rate	Exogenous range, fn( $\pi_i$ )	.02	.01	.023
В	Base income per agent	Exogenously fixed	5	0	50
$\pi_i$	Net income of the <i>i</i> th agent	Agent behavior			
G	Maximum aggregate gains from collective action	Realized value $(pG)$ depends on local Cooperator frequency	50	10	75
С	Labor cost of producing collective good, per Cooperator	Exogenously fixed	1	.5	1.8
р	Proportion of Cooperators in local group	Demographic and evolutionary	.33	0	1
т	Cooperator's cost of monitoring	Exogenously fixed	0.1	0	0.25
M	Manager's cost of monitoring	Exogenously fixed $(M \le m)$	0.1	0	0.25
е	Cooperator's cost of punishing Defectors	Exogenously fixed	0.5	0	2
Ε	Manager's cost of punishing Defectors	Exogenously fixed $(E \le e)$	0.5	0.25	1
S	Cost to Defector of being punished by a Cooperator	Exogenously fixed	.5	.1	1
S	Cost to Defector of being punished by a Manager	Exogenously fixed (S>>s)	3	2	5
γ	Fee paid to Manager by each Cooperator	Exogenously fixed	1	.5	1.5

## Table 4. Parameters for the Managerial Mutualism Model.

<sup>a</sup> Initial mean value; all parameter values are per period, unless noted otherwise.

		Mean termina	l proportion:	
			Patches with	 l
Parameter	Value	<b>Cooperators</b> a Manager		Comments
All default values	(see Table 4)	.88	1.00	Stable MM regime emerges in most runs
Mutation rate (µ)	Low (.001)	.89	1.00	Little effect compared to default settings
	High (.05)	.87	1.00	Little effect compared to default settings
Value of collective good (G)	Low (10)	.53	.74	Much less favorable for MM regime than default
	High (75)	.79	.88	Less favorable for MM regime than default
Production cost $(c)$	Low (.5)	.89	1.00	Little effect compared to default settings
	High (1.8)	.17	.21	Very unfavorable to MM regime
Management fee $(\gamma)$	Low (.5)	.39	.28	Very unfavorable to MM regime
C ()	High (1.5)	.44	.76	Very unfavorable to MM regime
Cost of monitoring:				,
for a Manager $(M)$	Low (0)	.86	.97	Little effect compared to default settings
	High (.25)	.75	.84	Less favorable for MM regime than default
for a Cooperator ( <i>m</i> )	Low (0)	.88	1.00	No effect compared to default settings
	High (.25)	.88	1.00	No effect compared to default settings
Cost of enforcement:				
for a Manager $(E)$	Low (.25)	.88	1.00	No effect compared to default settings
	High (1)	.01	.01	Very unfavorable to MM regime
for a Cooperator (e)	Low (0)	.88	1.0	No effect compared to default settings
	High (2)	.88	1.0	No effect compared to default settings
Cost of being punished:				
by a Cooperator (s)	Low (.1)	.88	1.0	No effect compared to default settings
	High (1)	.88	1.0	No effect compared to default settings
by a Manager (S)	Low (2)	.00	.01	Much less favorable for MM regime than default
	High (5)	.89	1.0	Little effect compared to default settings
Mortality rate	Low (.01)	.90	1.0	More favorable for MM regime than default
-	High (.023)	.42	.14	Very unfavorable to MM regime
Reproductive rate	Low (.02)	.17	.02	Very unfavorable to MM regime
	High (.05)	.90	1.0	Same as low-mortality setting
Maximum local group size ( <i>n</i> )	Low $(n = 5)$	.13	.42	Very unfavorable to MM regime
	High $(n = 15)$	.62	.82	Much less favorable for MM regime than default

 Table 5. Managerial Mutualism (MM) Outcomes (Means for Last 10 Periods over 100 Runs, 2000 Periods per Run).

## Figures

**Figure 1.** Patron-Client lattice at end of two runs (default parameter settings, 2,000 periods). (a) Run 1: Dove = 46.0 percent, Solo = 48.7 percent, Client = 5.2 percent, Patron = 0 percent. (b) Run 2 (Patron-Client equilibrium): Dove = 0 percent, Solo = .7 percent, Client = 64.5 percent, Patron = 34.9 percent.

Figure 2. Patron-Client lattice, shortly after initial all-Dove seeding.

**Figure 3.** Managerial mutualism model, showing agent payoffs with and without a manager present (given default parameter values). (a) In groups without a manager, defectors outcompete cooperators unless the latter are very common. (b) With a manager present, cooperators always do better than defectors, but the manager is at a disadvantage until defectors are in the minority.



# **(a)**

•	۰	٠		$\mathbb{P}_{i}$		٠	•	•	•
	٠	٠		٠	٠			٠	•
°0	٠	$\mathbb{R}^{2}$	٠	•••	•••	۰		•	•
•	٠	•	•	٠	•°°	٠	٠	٠	
•	٠	٠	$\mathbb{R}^{n}$	٠	٠			٠	°°°
<b>°</b>		•	•	٠	•	•	٠		٠
•		• °°°	•	•	•	•	•	•	•
	٠	• %°	• •		•				•
	•		• • •		•	•			•

**(b)** 

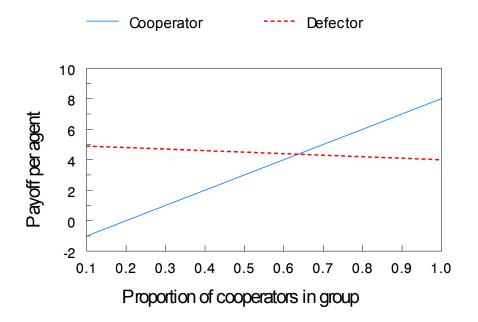
•	٠	•	•	٠	•	٠	•	٠	٠
•••		<b>*</b> **		٠	٠	٠	٠	$\mathbb{R}^{n}$	•
•	٠	•••					٠	٠	٠
•••	٠	•	٠	٠	٠		•	•	٠
٠	٠			٠	٠		•	٠	•
•			•	•		٠	٠	•••	•
•	•••		٠		٠		٠		٠
•	٠	٠	•	٠	••	•••		٠	٠
•••	٠	•••	•		٠	•		•	٠
٠	٠	۰		٠		•	•	•	٠

[Figure 2]

°°°	°0	°0	°0	°0	۰	°°	°°°	•	°°°
°°°	°°°	•	•	•	•	°0	°0	۰	°0
•••	•	°°	۰	•	°•	•	٥	•	•
•	•	°°	°0	°°°	°•	•	٥	°°°	°°
°°°	•	•	٥	°°°	°°°	°0	۰	۰	
°0	°0	°°	°0	°0	°0	$\mathbf{F}$	°°°	°°°	•
•••	•	$\mathbb{R}$	°0	•	°•	•	•	°•	•
•	•••	•	•	•	×	•		°0	•••
°0	°•	°•	$\mathbf{F}$	•	•	°•	•••	°•	°0
°0	°0	•	•	°°°	•	۰	$\mathbf{F}$	•	°0



(a)



(b)

