Other Runge Kutta Methods

- Runge Kutta Fehlberg
  - Computes y_{n+1} by 2 methods simultaneously
  - Different orders so h size is validated
    - 4^{th} and 5^{th}
  - Improvement on extrapolation
- Runge Kutta Merson
  - Variant 4^{th} order

An Algorithm for the Runge–Kutta–Fehlberg Method

$$k_1 = h \cdot f(x_n, y_n),$$
$$k_2 = h \cdot f\left(x_n + \frac{h}{4}, y_n + \frac{k_1}{4}\right),$$
$$k_3 = h \cdot f\left(x_n + \frac{3h}{8}, y_n + \frac{3k_1}{32} + \frac{9k_2}{32}\right),$$
$$k_4 = h \cdot f\left(x_n + \frac{12h}{13}, y_n + \frac{1932k_1}{2197} - \frac{7200k_2}{2197} + \frac{7296k_3}{2197}\right),$$
$$k_5 = h \cdot f\left(x_n + h, y_n + \frac{439k_1}{216} - 8k_2 + \frac{3680k_3}{513} - \frac{845k_4}{4104}\right),$$
$$k_6 = h \cdot f\left(x_n + \frac{h}{2}, y_n - \frac{8k_1}{27} + 2k_2 - \frac{3544k_3}{2565} + \frac{1859k_4}{4104} - \frac{11k_5}{40}\right);$$
$$\hat{y}_{n+1} = y_n + \left(\frac{25k_1}{216} + \frac{1408k_3}{2565} + \frac{2197k_4}{4104} - \frac{k_5}{5}\right), \text{ with global error } O(h^4),$$
$$y_{n+1} = y_n + \left(\frac{16k_1}{135} + \frac{6656k_3}{12825} + \frac{28561k_4}{56430} - \frac{9k_5}{50} + \frac{2k_6}{55}\right),$$
with global error $O(h^5)$;
$$\text{Error, } E \doteq \frac{k_1}{360} - \frac{128k_3}{4275} - \frac{2197k_4}{75240} + \frac{k_5}{50} + \frac{2k_6}{55}.$$

Runge Kutta Merson

$$k_1 = h \cdot f(x_n, y_n),$$
$$k_2 = h \cdot f\left(x_n + \frac{h}{3}, y_n + \frac{k_1}{3}\right),$$
$$k_3 = h \cdot f\left(x_n + \frac{h}{3}, y_n + \frac{k_1}{6} + \frac{k_2}{6}\right),$$
$$k_4 = h \cdot f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{8} + \frac{3k_3}{8}\right),$$
$$k_5 = h \cdot f\left(x_n + h, y_n + \frac{k_1}{2} - \frac{3k_3}{2} + 2k_4\right);$$
$$y_{n+1} = y_n + \frac{(k_1 + 4k_4 + k_5)}{6} + O(h^5);$$
$$\text{Error, } E \doteq \frac{1}{30}(2k_1 - 9k_3 + 8k_4 - k_5).$$

Comparison of Methods

Method	Global Error	Local Error	f(x,y) Evaluations
Euler	O(h)	$O(h^2)$	1
Modified Euler	$O(h^2)$	$O(h^3)$	2
Runge Kutta	$O(h^4)$	$O(h^5)$	4
Runge Kutta Merson	$O(h^4)$	$O(h^5)$	5
Runge Kutta Fehlberg	$O(h^5)$	$O(h^6)$	6

Multi-Step Methods

- Apply more information
  - Use more than just initial conditions
- Apply data from more previous iterations
- Combination of
  - Interpolation
  - Diff.EQ solving

Adam's Method (1)

Consider a rearrangement

$$\frac{dy}{dx} = f(x,y)$$

$$dy = f(x,y)dx$$

$$\int_{x_n}^{x_{n+1}} dy = y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} f(x,y)dx$$

Use an interpolation form

$$f(x,y) = \frac{1}{2}h^2(f_n - 2f_{n-1} + f_{n-2})x^2 + \frac{1}{2}h(3f_n - 4f_{n-1} + f_{n-2})x + f_n$$

$$\int_{x_n}^{x_{n+1}} f(x,y)dx = \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2})$$

Adam’s Method (2)

Final Form

$$y_{n+1} = y_n + \frac{h}{12}\left(23f_n - 16f_{n-1} + 5f_{n-2}\right) + O\left(h^4\right)$$

At higher orders

$$y_{n+1} = y_n + \frac{h}{24}\left(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right) + O\left(h^5\right)$$

Milne’s Method

Also include integration methods

$$\int_{x_{n-3}}^{x_{n+1}} dy = \int_{x_{n-3}}^{x_{n+1}} f(x, y)\,dx = \int_{x_{n-3}}^{x_{n+1}} P_2(x)\,dx$$

Working through the math

$$y_{n+1} = y_{n-3} + \frac{4h}{3}\left(2f_n - f_{n-1} + 2f_{n-2}\right) + O\left(h^5\right)$$

Poorer error overcome by prediction-correction

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left(\underline{f_{n+1}} + 4f_n + f_{n-1}\right) + O\left(h^5\right)$$

Use predicted y_{n+1}

Adams-Moulton Method

More stable than Milne using Undetermined Coefficients

Predict $y_{n+1} = y_n + \frac{h}{24}\left(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right) + O\left(h^5\right)$

Correct $y_{n+1} = y_n + \frac{h}{24}\left(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right) + O\left(h^5\right)$

Mid-step Interpolations (Nice for halving h)

$$y_{n-\frac{1}{2}} = \frac{1}{128}\left(35y_n + 140y_{n-1} - 70y_{n-2} + 28y_{n-3} - 5y_{n-4}\right)$$

$y_{n-\frac{3}{2}} = \frac{1}{\text{[illegible]}}\left(-y_n + 24y_{n-1} + 54y_{n-2} - 16y_{n-3} + 3y_{n-4}\right)$

Fewer f evaluations than Runge Kutta methods!

First steps require other methods!

Convergence Criteria (1)

- How small should h be?
- Definitions
  - y_p = predicted value of y_{n+1}
  - y_c = corrected value of y_{n+1}
  - $D = y_c - y_p$
  - N = number of decimal digits required
- Approach
  - Continue to correct corrections until convergence
  - Examine the math [not shown]

Convergence Criteria (2)

Adams-Moulton

Accuracy: $D \times 10^N < 14.2$

Criteria $\begin{cases} h < \dfrac{24/9}{|f_y|} \\ D\times 10^N < \dfrac{24/9}{h|f_n|} \end{cases}$

Milne

Accuracy: $D \times 10^N < 29$

Criteria $\begin{cases} h < \dfrac{3}{|f_y|} \\ D\times 10^N < \dfrac{3}{h|f_n|} \end{cases}$

Monitor the iterations to verify that D isn't too large!

Multi-Valued Methods

1. Start with 3rd-order Taylor approximation
2. Predict values for y derivatives by differentiating f
3. Correct predictions using secant information

- Bonus
  - Future steps are already partially computed