# Notes on field theory

## 1 Conventions

Unit: $\hbar = c = 1$, GeV $= 10^9 \times 1.6\ 10^{-19}$ J.

Metric: $\eta^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$

## 2 Scalar field theory

Action for a real scalar

$$S = \frac{1}{2}\int d^4x\,[(\partial_\mu\phi)^2 - m^2\phi^2] \tag{1}$$

Field equation:

$$(\Box + m^2)\phi = 0 \tag{2}$$

Expansion to plane-wave modes:

$$\phi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2E_\mathbf{k}}(a_\mathbf{k}e^{-ik\cdot x} + a^\dagger_\mathbf{k}e^{ik\cdot x}) \tag{3}$$

where $E_\mathbf{k} = (\mathbf{k}^2 + m^2)^{1/2}$.

Canonical quantization:

$$[a_\mathbf{k},\, a^\dagger_{\mathbf{k}'}] = (2\pi)^3 2E_\mathbf{k}\delta^3(\mathbf{k} - \mathbf{k}') \tag{4}$$

other commutators are zero.

Hamiltonian:

$$H = \int \frac{d\mathbf{k}}{(2\pi)^3 2E_\mathbf{k}}\, a^\dagger_\mathbf{k} a_\mathbf{k} + \mathrm{const} \tag{5}$$

The vacuum $|0\rangle$ is the ground state of $H$: $a_\mathbf{k}|0\rangle = 0$. The Fock space is constructed by applying creation operators on $|0\rangle$:

$$a^\dagger_k|0\rangle = |\mathbf{k}\rangle, \qquad a^\dagger_{\mathbf{k}_1}a^\dagger_{\mathbf{k}_2} = |\mathbf{k}_1,\, \mathbf{k}_2\rangle \tag{6}$$

These are one-particle, two-particle,..., states.

Complex scalar field theory:

$$S = \int d^4x\,[|\partial_\mu\phi|^2 - m^2|\phi|^2] \tag{7}$$

where $|\phi|^2 = \phi^*\phi$. Plane-wave expansion

$$\phi(x) = \int \frac{d\mathbf{k}}{(2\pi)^3 2E_{\mathbf{k}}} (a_{\mathbf{k}} e^{-ik\cdot x} + b^\dagger_{\mathbf{k}} e^{ik\cdot x}) \tag{8}$$

where $[a_{\mathbf{k}},\, a^\dagger_{\mathbf{k}'}] = (2\pi)^3 2E_{\mathbf{k}} \delta^3(\mathbf{k}-\mathbf{k}')$, $[b_{\mathbf{k}},\, b^\dagger_{\mathbf{k}'}] = (2\pi)^3 2E_{\mathbf{k}} \delta^3(\mathbf{k}-\mathbf{k}')$, other commutators are zero.

**Noether theorem** Assume the Lagrangian is invariant under a continous symmetry:

$$\mathcal{L}(\phi + \epsilon\delta\phi) = \mathcal{L}(\phi) \tag{9}$$

then there exist a conserved current,

$$\partial_\mu j^\mu = 0, \qquad j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \tag{10}$$

In the case of the complex scalar field, the symmetry is $\phi \to e^{-i\epsilon}\phi$, or $\delta\phi = -i\phi$, $\delta\phi^* = -i\phi^*$. The conserved current is

$$j^\mu = i(\phi^* \partial^\mu \phi - \partial^\mu \phi^* \phi) \tag{11}$$

the conserved charge is

$$Q = \int d^3\mathbf{x}\, j^0 = \int \frac{d\mathbf{k}}{(2\pi)^3 2E_{\mathbf{k}}} \, (a^\dagger_{\mathbf{k}} a_{\mathbf{k}} - b^\dagger_{\mathbf{k}} b_{\mathbf{k}}) \tag{12}$$

The operator $a^\dagger_{\mathbf{k}}$ increases $Q$ by one unit, while $b^\dagger_{\mathbf{k}}$ decreases $Q$ by one unit. One can think about $b^\dagger_{\mathbf{k}}$ as the operator that creates an antiparticle, while $a^\dagger_{\mathbf{k}}$ creates a particle.

# 3 Spin-1 fields

## 3.1 Massless spin-1 fields

An example of a massless spin-1 field is the electromagnetic field (photons). It is described by a 4-vector $A_\mu$. Its components are the scalar potential and the vector potential, $A_\mu = (\phi, -\mathbf{A})$. The action is

$$S = -\frac{1}{4} \int d^4x \, F^{\mu\nu} F_{\mu\nu} \tag{13}$$

where $F_{\mu\nu}$ is the field strength tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{14}$$

The field equation is

$$\partial_\mu F^{\mu\nu} = 0 \tag{15}$$

These constitute two of the four Maxwell's equations. The other two are the Bianchi identity, $\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$.

$F_{\mu\nu}$ is invariant under gauge transformations,

$$A_\mu \to A_\mu + \partial_\mu \alpha \tag{16}$$

By chosing an appropriate $\alpha$, one can impose the Lorentz gauge condition $\partial_\mu A^\mu = 0$. The field equation then becomes

$$\Box A_\nu = 0 \tag{17}$$

and the plane-wave solution is

$$A_\mu = \epsilon_\mu(p) e^{-ip\cdot x} \tag{18}$$

where $p^2 = 0$. Here $\epsilon_\mu$ is a polarization vector.

The Lorentz gauge condition requires that $p \cdot \epsilon = 0$. This means that there are only three independent polarization vectors possible. Even after imposing the Lorentz gauge, one still can make a residual gauge tranformation $A_\mu \to A_\mu + \partial_\mu \alpha$ where $\Box \alpha = 0$. This amounts to changing $\epsilon_\mu \to \epsilon_\mu + a p_\mu$ with arbitrary constant $a$. Thus one has to impose one additional constraint to fix $\epsilon_\mu$. For example, one can impose $\epsilon_0 = 0$. The $\epsilon_\mu$ is a spatial vector perpendicular to $\mathbf{p}$. For example, if $p = (p, 0, 0, p)$ then there are two independent polarization vectors: $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$. These correspond to linearly polarized plane waves. Circularly polarized waves corresponds to $\epsilon_\mu = (0, 1, i, 0)/\sqrt{2}$ or $(0, 1, -i, 0)/\sqrt{2}$.

Denote the two independent polarizations as $\epsilon^\lambda_\mu(p)$, $\lambda = 1, 2$. We can normalize the vectors so that

$$\epsilon^\lambda \cdot \epsilon^{\lambda'} = -\delta^{\lambda\lambda'} \tag{19}$$

The quantum field $A_\mu$ can then be expanded as

$$A_\mu(x) = \int \frac{d\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}} (a_{\mathbf{p}\lambda} \epsilon^\lambda_\mu(p) e^{-ip\cdot x} + a^\dagger_{p\lambda} \epsilon^{\lambda *}_\mu(p) e^{ip\cdot x}) \tag{20}$$

Quantization of the Maxwell's theory leads to the commutation relation

$$[a_{\mathbf{p}\lambda},\, a^\dagger_{p'\lambda'}] = (2\pi)^3 2E_\mathbf{p} \delta_{\lambda\lambda'} \delta(\mathbf{p} - \mathbf{p}') \tag{21}$$

So $a^\dagger_{\mathbf{p}\lambda}|0\rangle$ is the state with one photon with momentum $\mathbf{p}$ and polarization $\epsilon^\lambda$.

## 3.2 Massive spin-1 field

Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu \tag{22}$$

The action is no longer gauge-invariant.

The field equation is

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0 \tag{23}$$

Taking $\partial_\nu$ of that equation, we find $\partial_\nu A^\nu = 0$. Hence the field equation is $(\Box + m^2) A_\mu = 0$. The plane-wave solution is

$$A_\mu = \epsilon_\mu e^{-ip\cdot x} \tag{24}$$

where $p^2 = m^2$ and $p \cdot \epsilon = 0$. There is 3 possible polarization vectors $\epsilon$. A massive spin-1 particle has 3 possible polarizations.

# 4 Spin-1/2 fields

## 4.1 Lorentz algebra

Recall the operators of rotations (angular momenta) satisfy the commutation relations of a SU(2) group:

$$[J^i,\, J^j] = i\epsilon^{ijk} J^k \tag{25}$$

The tensor representation of $J^i$ is $J^i = \sigma^i/2$, where $\sigma^i$ are the Pauli matrices.

Instead of $J^1$, $J^2$, $J^3$, we shall use $J^{ij}$ so that $J^{ij} = -J^{ji}$, $J^{12} \equiv J^3$, $J^{23} \equiv J^1$, $J^{31} \equiv J^2$. The commutators of $J^{ij}$ can be written as

$$[J^{ij},\, J^{kl}] = i(\delta^{ik} J^{jl} + \delta^{jl} J^{ik} - \delta^{il} J^{jk} - \delta^{jk} J^{il}) \tag{26}$$

(check!). This commutation relation can also be found if one recalls that, for orbital moment, $J^{ij} = x^i p^j - x^j p^i$, and $[x^i,\, p^j] = i\delta^{ij}$.

The commutators of the Lorentz group is a direct generalization:

$$[J^{\mu\nu},\, J^{\alpha\beta}] = -i(\eta^{\mu\alpha} J^{\nu\beta} + \eta^{\nu\beta} J^{\mu\alpha} - \eta^{\mu\beta} J^{\nu\alpha} - \eta^{\nu\alpha} J^{\mu\beta}) \tag{27}$$

It is instructive to write these commutators in components. Introducing $J^1 \equiv J^{23}$, etc., and $K^i \equiv J^{0i}$, we have

$$[J^i,\, J^j] = \quad i\epsilon^{ijk} J^k, \tag{28}$$

$$[J^i,\, K^j] = \quad i\epsilon^{ijk} K^k, \tag{29}$$

$$[K^i,\, K^j] = -i\epsilon^{ijk} J^k \tag{30}$$

Introducing

$$L^i_\pm = \frac{J^i \pm K^i}{2} \tag{31}$$

the Lorentz algebra decomposes into two SU(2) algebras,

$$[L^i_+,\, L^j_+] = i\epsilon^{ijk} L^k_+, \qquad [L^i_-,\, L^j_-] = i\epsilon^{ijk} L^k_- \tag{32}$$

The tensor representation of the Lorentz group is equivalent to

$$\vec{L}_+ = \frac{1}{2}\begin{pmatrix} \vec\sigma & 0 \\ 0 & 0 \end{pmatrix}, \qquad \vec{L}_- = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \vec\sigma \end{pmatrix}, \tag{33}$$

which means

$$\vec{J} = \frac{1}{2}\begin{pmatrix} \vec\sigma & 0 \\ 0 & \vec\sigma \end{pmatrix}, \qquad \vec{K} = \frac{1}{2}\begin{pmatrix} -i\vec\sigma & 0 \\ 0 & i\vec\sigma \end{pmatrix}, \tag{34}$$

## 4.2 Dirac matrices

The Dirac matrices $\gamma^\mu$ are $4\times$ basis that satisfy the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{35}$$

In the chiral basis they are chosen as

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{36}$$

One also introduces

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{37}$$

In the spinor representation of the Lorentz algebra, $J^{\mu\nu} = \sigma^{\mu\nu}$. (check!).

A Dirac spinor is a four-component spinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{38}$$

One defines $\bar{\psi} = \psi^\dagger\gamma^0$.

## 4.3 The field theory for spin-1/2 particles

Dirac Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \tag{39}$$

Plane wave expansion:

$$\psi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3(2E_p)}(a_{\mathbf{p}s}u_s(p)e^{-ip\cdot x} + b^\dagger_{\mathbf{p}s}v_s(p)e^{ip\cdot x}) \tag{40}$$

Again in this equation it is implied that $p_0 = +E_{\mathbf{p}}$. The spinor $u_s(p)$ and $v_s(p)$ satisfy the equations

$$(\not{p} - m)u_s(p) = 0 \tag{41}$$
$$(\not{p} + m)v_s(p) = 0 \tag{42}$$

and are normalized so that

$$u^\dagger_s(p)u_r(p) = 2E_{\mathbf{p}}\delta_{sr}, \tag{43}$$
$$v^\dagger_s(p)v_r(p) = 2E_{\mathbf{p}}\delta_{sr} \tag{44}$$

One choice is

$$u_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi_s \\ \sqrt{p \cdot \bar{\sigma}}\xi_s \end{pmatrix} \tag{45}$$

with $\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Analogously one can chose

$$v_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi_s \\ -\sqrt{p \cdot \bar{\sigma}}\xi_s \end{pmatrix} \tag{46}$$

. One can check that

$$\bar{u}_s u_r = \quad 2m\delta_{sr}, \tag{47}$$

$$\bar{v}_s v_r = -2m\delta_{sr} \tag{48}$$

The operators $a^\dagger_{\mathbf{p}}$ and $b^\dagger_{\mathbf{p}}$ create a particle and an antiparticle, respectively. In the case of electrons, the antiparticle is called the positron.

# 5 Renormalizability

In the unit system $\hbar = c = 1$, all dimensions are powers of energy. A quantity $O$ is said to have dimension $\Delta$ if it is measured in unit of $(\mathrm{GeV})^\Delta$. We write in this case $[O] = \Delta$.

The action is dimensionless ($\hbar = 1$), and since $S = \int d^4x\, \mathcal{L}$ and $[x] = -1$, $[\mathcal{L}] = 4$.

The Standard Model Lagrangian is renormalizable. The renomalizability condition requires that there are only operators with dimensions less or equal to 4 in the Lagrangian. In other words, the coefficients in front of the operators have non-negative dimensions.

In a scalar field theory the only allow interaction terms are cubic ($\phi^3$) or quartic ($\phi^4$) in fields. In a theory with fermions and scalars, the only interaction term between a boson and a fermion is $\bar{\psi}\psi\phi$, $i\bar{\psi}\gamma^5\psi\phi$. For example, $\bar{\psi}\gamma^\mu\psi\partial_\mu\phi$ has dimension 5 and is not renormalizable (irrelevant).

The justification for renormalizability is that if nonrenormalizable terms are present in the Lagrangian, the coefficient in front of them would be proportional to $1/M^{\Delta-4}$, where $\Delta$ is the dimension of the operatore and $M$ is some energy scale characteristic of the physics beyond the Standard Model. We now experimentally that $M$ is high, therefore these terms can be neglected.

For a coupling between a fermion and a gauge field, the term $\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}$ is gauge invariant, but not renormalizable (dimension 5). The term $\bar{\psi}\gamma^\mu\psi A_\mu$ is renormalizable. How to preserve gauge invariance is the subject of next Section.

# 6 Gauge-invariant interactions

The Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi \tag{49}$$

is invariant under phase rotations

$$\psi \to \psi' = e^{i\alpha}\psi \tag{50}$$

where $\alpha$ is a constant. But if we generalize it to spatially varying $\alpha$,

$$\psi \to \psi' = e^{i\alpha(x)}\psi \tag{51}$$

The reason is that $\partial_\mu\psi$ does not tranform homogeneously,

$$\partial_\mu\psi \to \partial_\mu\psi' = e^{i\alpha}(\partial_\mu\psi + i\partial_\mu\alpha\psi) \tag{52}$$

It is possible to correct for this by introducing a gauge field $A_\mu$ into the theory. We replace the derivative $\partial_\mu$ by the covariant derivative $D_\mu$, defined as

$$D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi \tag{53}$$

If we requires that under (51) with $\alpha = \alpha(x)$, $A_\mu$ simultaneous undergoes a gauge tranformation

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\alpha(x) \tag{54}$$

then

$$D_\mu\psi \to (D_\mu\psi)' = \partial_\mu\psi' + ieA'_\mu\psi' = e^{i\alpha(x)}D_\mu\psi \tag{55}$$

Hence the following Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F^2_{\mu\nu} \tag{56}$$

is invariant under gauge tranformations (51), (54). This is the Lagrangian of quantum electrodynamics, describing electrons and photons.

# 7 Nonabelian gauge theories

The gauge theory described in the previous Section is called U(1) gauge theory, taking its name from the group of phase rotations (50). Let us generalize it to the SU(2) group. Consider a doublet of fermion fields

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{57}$$

The Lagrangian $\bar{\psi}(i \not\partial - m)\psi$ (same mass for $\psi_1$ and $\psi_2$) is invariant under SU(2) rotations

$$\psi \to \psi' = U\psi \tag{58}$$

where $U$ is a spatial unitary matrix, $UU^\dagger = U \dagger U = 1$, $\det U = 1$ (it is also invariant under U(1) phase rotation $\psi \to e^{i\alpha}\psi$, which we will not consider here). In order to "promote" this symmetry to a gauge symmetry with $U = U(x)$, we introduce a gauge field $2\times 2$ matrix $A_\mu$, and write

$$D_\mu\psi = \partial_\mu - igA_\mu\psi \tag{59}$$

We require that under the transformation $\psi \to \psi' = U(x)\psi$ the convariant derivative tranforms homogeneously,

$$D_\mu \psi \to (\partial_\mu - igA'_\mu)\psi' = U(\partial_\mu - igA_\mu \psi) \tag{60}$$

That fixes the tranformation law for $A_\mu$:

$$A_\mu \to A'_\mu = UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \tag{61}$$

It is possible to restrict $A_\mu$ to be a Hermitian, traceless matrix. This property is preserved under gauge tranformation (61). That means we have 3 components of $A$,

$$A_\mu = A^a_\mu \frac{\tau^a}{2} \tag{62}$$

Thus the coupling between fermions and the gauge field can be introduced in a gauge-invariant manner,

$$L = \bar{\psi}\left(\partial_\mu - igA^a_\mu \frac{\tau^a}{2}\right)\psi - m\bar{\psi}\psi \tag{63}$$

We still need a gauge-invariant action for the gauge field alone, similar to the Maxwell action. Notice that

$$D_\mu D_\nu \psi - D_\nu D_\mu \psi = -igF_{\mu\nu}\psi \tag{64}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{65}$$

it follows that under gauge transformation $F_{\mu\nu} \to UF_{\mu\nu}U^{-1}$. Thus we can add to the Lagrangian the Yang-Mills Lagrangian

$$-\frac{1}{2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} \tag{66}$$

which is invariant under gauge tranformations. In components

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu \tag{67}$$

The term $-\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}$ contains both the kinetic terms of the gauge field and the interaction terms.