Notes on the Standard Model Lagrangian

1 The gauge group

The gauge group of the Standard Model is SU(3)×SU(2)×U(1). We don't know why it is this group, but that is is how Nature is.

The gauge part is therefore

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}G^{\alpha}_{\mu\nu}G^{\alpha}_{\mu\nu} - \frac{1}{4}F^{a}_{\mu\nu}F^{a}_{\mu\nu} - \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 \tag{1}$$

where $\alpha = 1 \ldots 8$, $a = 1 \ldots 3$. The $G^{\alpha}_{\mu\nu}$ is the field strength of the SU(3) gauge field; $F^{a}_{\mu\nu}$ is the SU(2) field tensor; and B_μ is the

The nomenclature is as follows: the SU(3) gauge field mediates strong interaction and is called the gluon field, the SU(2) gauge group is called weak isospin, and the U(1) is called weak hypercharge (shortened to hypercharge). The SU(2)×U(1) gauge group is resposible for the electroweak (electromagnetic and weak) interactions.

2 The Higgs sector

The SU(3) gauge symmetry is unbroken, but the SU(2)×U(1) gauge symmetry is spontaneously broken down to U(1) group. As the result, three of the electroweak gauge bosons become massive (the $W^\pm$ and Z^0 boson), and one remains massless (the photon). The 8 gluons mediating strong interaction remains massless (this is in fact more complicated by confinement, but we ignore this subtlety here).

In the Standard Model, the breaking of the electroweak symmetry is accomplished by the Higgs field

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{2}$$

The Higgs field is in the doublet representation (**2**) of SU(2) and is assigned hypercharge $Y = 1/2$. The Lagrangian of the Higgs field is

$$\mathcal{L} = D_\mu\Phi^\dagger D_\mu\Phi - \lambda\left(\Phi^\dagger\Phi - \frac{v^2}{2}\right)^2 \tag{3}$$

The covariant derivative of the Higgs field is defined as

$$D_\mu\Phi = \left(\partial_\mu - ig\frac{\tau^a}{2}A^a_\mu - i\frac{1}{2}g' B_\mu\right)\Phi \tag{4}$$

The factor $\frac{1}{2}$ in front of g' is the hypercharge of the Higgs field. It is clear that the hypercharge assigment is somewhat arbitrary, since it can be absorbed into the definition of g'. However, once the hypercharge of the Higgs field is fixed, the hypercharges of other fields are fixed.

The Higgs potential has degenerate minima at $\Phi^\dagger\Phi = |\phi^+|^2 + |\phi^0|^2 = v^2/2$. Using gauge symmetries, it is always possible to rotate Φ so that $\phi^+ = 0$ and ϕ^0 is real. We thus expand

$$\Phi = \begin{pmatrix} 0 \\ \frac{v+H}{\sqrt{2}} \end{pmatrix} \tag{5}$$

Ignoring for a moment the gauge fields, the Lagrangian for H has the form

$$L = \frac{1}{2}(\partial_\mu H)^2 - \lambda v^2 H^2 - \lambda v H^3 - \frac{\lambda}{4} \tag{6}$$

The mass of the quantum of the H field, called the Higgs particle, is $m_H = \sqrt{2\lambda}\, v$

3 The gauge boson sector

The quadratic part of the gauge boson Lagrangian is

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \frac{g^2 v^2}{8}(A^1_\mu A^1_\mu + A^2_\mu A^2_\mu) + \frac{v^2}{8}(-gA^3_\mu + g' B_\mu)^2 \tag{7}$$

We find that $A^{1,2}_\mu$ obtain masses equal to $m_W = gv/2$. The Lagrangian is, however, not diagonal in the A^3_μ, B_μ variables. To diagonalize the Lagrangian we introduce new fields,

$$Z_\mu = \cos\theta_W A^3_\mu - \sin\theta_W B_\mu \tag{8}$$
$$A_\mu = \sin\theta_W A^3_\mu + \cos\theta_W B_\mu \tag{9}$$

where the Weinberg angle is defined so that

$$\tan\theta_W = \frac{g'}{g} \tag{10}$$

The quadratic Lagrangian for the new fields is

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)^2 + \frac{m_Z^2}{2} Z_\mu Z_\mu - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \tag{11}$$

where

$$m_Z = \frac{1}{2}\sqrt{g^2 + g'^2}\, v = \frac{m_W}{\cos\theta_W} \tag{12}$$

Therefore the Z_μ field is massive, and corresponds to the Z boson. The A_μ field remains massless, and corresponds to the photon. The Standard Model unifies the weak interactions, mediated by the W and Z bosons, with the electromagnetic interactions, mediated by the photon.

The Z boson is heavier than the W boson. From the ratio of their masses one can determine the Weinberg angle

$$\frac{m_W}{m_Z} = \cos\theta_W \tag{13}$$

Taking the experimental values of the masses, $m_Z = 91.2$ GeV and $m_W = 80.4$ GeV, one finds $\sin^2\theta_W \approx 0.22$.

The Lagrangian also contains terms describing the interactions between Higgs boson and the gauge bosons. It is straightforward to write them down.

4 Fermions

The coupling of fermions to gauge bosons is accomplished by replacing, in the fermion Lagrangian, the derivative ∂_μ by the covariant derivative,

$$D_\mu = \partial_m u - igA_\mu^a T^a - ig' B_\mu Y \tag{14}$$

where T^a is the generators of the SU(2) algebra, which depend on which representation the fermions belong to. If the fermion belong to an SU(2) doublet, then $T^a = \tau^a/2$; if it is a SU(2) singlet then $T^a = 0$. Y is the hypercharge of the fermion.

In the standard models, left- and right-handed components of fermions interact differently with the gauge fields. To see that it is possible, recall that the current $\bar\psi\gamma^\mu\psi$ that couples to gauge fields can be written as a sum of a left-handed part and a right-handed part. Until masses are introduced the left- and right-handed part of a fermion do not "talk" to each other.

Rewriting the gauge fields in terms of $W_\mu^\pm$, Z_μ and A_μ, the interactions of fermions with gauge fields are described by the following terms in the Lagrangian,

$$\frac{g}{\sqrt{2}}(W_\mu^+ J_\mu^- + W_\mu^- J_\mu^+) + \frac{g}{\cos\theta_W} Z_\mu J_\mu^{\rm nc} + eA_\mu J_\mu^{\rm em} \tag{15}$$

where $J_\mu^\pm$, $J_\mu^{\rm nc}$ are called charge currents and neutral current (the current interacting with photons, $J_\mu^{\rm em}$), is the electromagnetic current). They are written as

$$J_\mu^- = \bar\nu\gamma_\mu\frac{1-\gamma^5}{2}e + \cdots + \bar u\gamma_\mu\frac{1-\gamma^5}{2}d + \cdots \tag{16}$$

and

$$J_\mu^{\rm nc} = \sum_f \bar f(T^3 - Q\sin^2\theta_W)f \tag{17}$$

where the sum over f is taken over all fermions, with left- and right-handed components counted separately.

The usual Dirac mass terms for fermions, $m\bar\psi\psi$, are forbidden by the electroweak SU(2)×U(1) symmetry. In the Standard Model, the fermions obtains masses from the Higgs field. If one had only one generations, the mass terms for the lower components of weak multiplets are

$$-y_e(\bar l_L\Phi)e_R \equiv (\bar\nu\phi^+ + \bar e_L\phi^0)e_R, -y_d \quad (\bar q_L\Phi)d_R \tag{18}$$

(plus complex conjugate). Expanding Φ into the condensate and the Higgs field, these terms generate the mass $m_e = y_e v/\sqrt{2}$ and $m_d = y_d v/\sqrt{2}$. To generate a mass term for the u quark, one uses the conjugate Higgs field, arraged in a SU(2) doublet:

$$\tilde{\Phi} = i\sigma_2 \Phi^* = \begin{pmatrix} \phi^{0*} \\ \phi^{+*} \end{pmatrix} \tag{19}$$

so the mass term for the u quark is

$$-y_u(\bar{d}_R \tilde{\Phi}) d_L + \text{h.c.} \tag{20}$$

and generate $m_u = y_u v/\sqrt{2}$. The mass terms also generate the interactionof the Higgs boson with the fermions, $-(m_f/v)\bar{f}fH$.

5 The CKM matrix